1.43 Proposition. (a) *For all $x, w, y \in X$ and for real z with $0 \le z \le \mathsf{r}(x,y)$ one has*

$$F(x,y|z) \ge F(x,w|z)\,F(w,y|z)\,.$$

(b) *Suppose that in the graph $\Gamma(P)$ of the Markov chain (X,P), the state w is a cut point between x and $y \in X$. Then*

$$F(x,y|z) = F(x,w|z)\,F(w,y|z)$$

for all $z \in \mathbb{C}$ with $|z| < \mathsf{s}(x,y)$ and for $z = \mathsf{s}(x,y)$.

Proof. (a) We have

$$\begin{aligned} p^{(n)}(x,y) &= \mathsf{Pr}_x[Z_n = y] \\ &\ge \mathsf{Pr}_x[Z_n = y\,,\ \boldsymbol{s}^w \le n] = \sum_{k=0}^{n} \mathsf{Pr}_x[Z_n = y\,,\ \boldsymbol{s}^w = k] \\ &= \sum_{k=0}^{n} \mathsf{Pr}_x[\boldsymbol{s}^w = k]\ \mathsf{Pr}_x[Z_n = |\ \boldsymbol{s}^w = k] = \sum_{k=0}^{n} f^{(k)}(x,w)\,p^{(n-k)}(w,y)\,. \end{aligned}$$

The inequality of statement (a) is true when $F(x,w|\cdot) \equiv 0.$ or $F(w,y|\cdot) \equiv 0$. So let us suppose that there are k and l such that $p^{(k)}(x,w) \ge f^{(k)}(x,w) > 0$ and $p^{(l)}(w,y) \ge f^{(l)}(w,y) > 0$ for some k and l. Then the general inequality $p^{(m+n)}(x,y) \ge p^{(m)}(x,w)p^{(n)}(w,y)$ implies that

$$\mathsf{r}(x,y) \le \min\{\mathsf{r}(x,w), \mathsf{r}(w,y), \mathsf{r}(y,y), \mathsf{s}(x,y)\}\,.$$

The product formula for power series now yields

$$\begin{aligned} G(x,y|z) &= f^{(n)}(x,y)\,z^n \\ &\ge \sum_{k=0}^{n} f^{(k)}(x,w)\,z^k\ p^{(n-k)}(w,y)\,z^{n-k} = F(x,w|z)G(w,y|z) \end{aligned}$$

for all real z with $0 \le z < \mathsf{r}(x,y)$. We can divide both sides of the last inequality by $G(y,y|z)$, and Theorem 1.38(b) implies the result for all z with $0 \le z < \mathsf{r}(x,y)$. Since we have power series with non-negative coefficients, we can let $z \to \mathsf{r}(x,y)$ from below to see that statement (a) also holds for $z = \mathsf{r}(x,y)$, regardless of whether the series converge or diverge at that point.

(b) We may suppose $w \ne y$. If w is a cut point between x and y, then the Markov chain must visit w before it can reach y. That is, $\boldsymbol{s}^w \le \boldsymbol{s}^y$, given that $Z_0 = x$. Therefore the strong Markov property yields

$$\begin{aligned} f^{(n)}(x,y) &= \mathsf{Pr}_x[\boldsymbol{s}^y = n] = \mathsf{Pr}_x[\boldsymbol{s}^y = n\,,\ \boldsymbol{s}^w \le n] \\ &= \sum_{k=0}^{n} f^{(k)}(x,w)\,f^{(n-k)}(w,y)\,. \end{aligned}$$

Here we have used that

$$\mathsf{Pr}_x[\boldsymbol{s}^y = n \mid \boldsymbol{s}^w = k] = \mathsf{Pr}_x[Z_n = y, Z_j \neq y(j = k, \ldots, n) \mid \boldsymbol{s}^w = k],$$

because the condition $\boldsymbol{s}^w = k$ comprises that $Z_j \neq y$ for $j_0, \ldots k-1$. We can now argue precisely as in the proof of (a), and the product formula for power series yields statement (b) for all $z \in \mathbb{C}$ with $|z| < \mathsf{s}(x,y)$ as well as for $z = \mathsf{s}(x,y)$. □

1.44 Exercise. Show that for distinct $x, y \in X$ and for real z with $0 \leq z \leq \mathsf{s}(x,x)$ one has

$$U(x,x|z) \geq F(x,y|z)\,F(y,x|z)\,.$$ □