

STAND-ALONE PROJECT

FINAL REPORT

P24028-N18

Project number

Project title **Hyperbolic structures in stochastics,
graph theory, and topology**

Project leader **Univ.-Prof. Dipl.-Ing. Dr. Wolfgang Woess**

Project website **<https://www.math.tugraz.at/~woess/#fwf>
<https://www.math.tugraz.at/~woess/FWF-P24028publ.html>**

Table of Contents

I. Summary for public relations work (German & English)

1. Zusammenfassung für die Öffentlichkeitsarbeit
2. Summary for public relations work

II. Brief project report

1. Report on research work
 - 1.1 *Statement on the overall impact of this project*
 - 1.2 *Information on the development of the research project*
 - 1.3 *Most important results and brief description of their significance*
 - 1.4 *Information on the execution of the project, use of available funds and (where appropriate) any changes to the original project plan*
2. Personnel development –
Importance of the project for the research careers of those involved
3. Other important aspects
 - 3.1 *Project-related participation in national and international conferences, seminar talks*
 - 3.2 *Organisation of symposia and conferences*
 - 3.3 *Funding principles*
 - 3.4 *International collaboration partners*
 - 3.5 *Doctoral Program “Discrete Mathematics”*

III. Attachments

1. Project publications
 - 1.1 *Peer-reviewed publications / already published*
 - 1.2 *Peer-reviewed publications / to appear*
 - 1.3 *Preprints / submitted*
 - 1.4 *PhD Theses*
2. Development of collaborations
3. Development of human resources in the course of the project
4. Applications for follow-up projects

IV. Cooperation with the FWF

I. Summary for public relations work

1. Zusammenfassung für die Öffentlichkeitsarbeit

Dieses mathematische Forschungsprojekt hat sich mit Fragestellungen befasst, die in verschiedener Weise eine „hyperbolische“ Natur haben. Dies ist im geometrischen Sinn zu verstehen: Vorbild ist die hyperbolische Ebene, wie sie im 19. Jahrhundert von Lobatschewski, Bolyai, sowie Gauss „entdeckt“ wurde. Sie ist negativ gekrümmt, und das klassische Parallelenaxiom gilt nicht mehr. Aus ihr leiten sich andere, hier auch diskrete Strukturen ab, nämlich Graphen und Gruppen mit hyperbolischen Eigenschaften. Graphen kann man sich als Netzwerke vorstellen; hier haben sie unendlich viele Knoten, die durch Kanten verbunden sind. Das typischste Graphen-Gegenstück zur hyperbolischen Ebene ist ein regulärer Baum. Gruppen kann man als Mengen von Symmetrieabbildungen von geometrischen Strukturen verstehen. Die Themen dieses Forschungsprojekts beziehen sich einerseits auf die Struktur von Graphen, Gruppen und metrischen Räumen mit „hyperbolischer“ Natur, und andererseits auf Zufallsprozesse, die in solchen Strukturen stattfinden: Markovprozesse, „Random Walks“, die zugeordnete Analysis und Potentialtheorie.

Die Themen dieses Projektes waren breit angelegt, und hatten insbesondere das Ziel, jungen ForscherInnen Entfaltungsmöglichkeiten zu geben und die lebhaftere Forschungsgruppe von Woess weiterzuentwickeln. In der Tat gab es insgesamt 6 ProjektmitarbeiterInnen, deren Aktivität im Projekt sie zum großen Teil zu anderen akademischen Stellen geführt hat, und eine große Zahl an Projektpublikationen, die in diesem Projekt begonnen, durchgeführt, bzw. beendet wurden.

Die Hauptresultate betreffen folgende Themen:

- Charakterisierung des Poissonrandes für transiente (hyperbolische) „Random Walks“ auf ebenen Quadrat-Pflasterungen;
- algebraische und dynamische Eigenschaften von Automaten-Gruppen, also Symmetriegruppen von regulären Wurzelbäumen, die durch Mealy-Automaten erzeugt werden;
- Brownsche Bewegung und harmonische Funktionen auf „treebolic spaces“ (horozyklische Produkte der hyperbolischen Ebene mit regulären Bäumen, baumartige Auffaltungen der hyperbolischen Ebene);
- Charakterisierung des Poissonrandes für „Random Walks“ auf nichtmittelbaren Baumslag-Solitar-Gruppen (das sind Symmetriegruppen von Varianten der „treebolic spaces“);
- asymptotische Analyse von isotropen Markovschen Zufallsprozessen auf ultrametrischen Räumen (den geometrischen Rändern von Bäumen);
- Untersuchung von Transienz und Rekurrenz von reflektierenden „Random Walks“ im mehrdimensionalen Raum.

2. Summary for public relations work

This mathematical research project was concerned with questions which are of “hyperbolic” nature in different ways. This is to be understood in terms of geometry: paradigm is the hyperbolic plane, as it has been “discovered” in the 19th century by Lobatschevski, Bolyai, as well as Gauss. It is negatively curved, and the classical parallel axiom is no more valid. It has led to other, here also discrete structures, that is, graphs and groups with hyperbolic properties. Graphs can be imagined as networks; here, they have infinitely many nodes which are connected by bonds (edges). The most typical graph-counterpart of hyperbolic plane is a regular tree. Groups can be understood as symmetry mappings of geometric structures. The topics of this research project concern on one hand the structure of graphs, groups and metric spaces with „hyperbolic“ nature, and on the other hand random processes which take place in such structures: Markov processes, “Random Walks“, the associated analysis and potential theory.

The topics of this project were laid out broadly and had, in particular, the aim to provide younger researchers with a fruitful environment and to further develop the vivid research group of Woess. In fact, there have been altogether 6 project collaborators, whose project activities have in large part led them to subsequent academic positions. A large number of project publications was begun, carried out, resp. completed within this project.

The main results concern the following topics: • characterisation of the Poisson boundary of transient (hyperbolic) random walks on plane square tilings; • algebraic and dynamical properties of automata groups, that is, symmetry groups of regular rooted trees which are generated by Mealy-automata; • Brownian motion and harmonic functions on “treebolic” spaces (horocyclic products of the hyperbolic plane with regular trees, tree-like unfoldings of hyperbolic plane); • characterisation of the Poisson boundary for random walks on non-amenable Baumslag-Solitar groups (symmetry groups of variants of the “treebolic” spaces); • asymptotic analysis of isotropic Markovian random processes on ultra-metric spaces (the geometric boundaries of trees); • investigation of transience and recurrence of reflected random walks in multi-dimensional space.

II. Brief project report

1. Report on the research work

1.1 Statement on the overall impact of this project

The following lines take up part of a text presented in relation with the previous FWF project P-19115-N18.

When I came to TU Graz in 1999, the small department of which I became the head was equipped with 2 assistant positions, both held by tenured persons working in quite different areas than myself. In the meantime, we have been merged with the Institute of Combinatorics and Optimisation to become the bigger Institute of Discrete Mathematics. The two working groups corresponding to my former institute have three regular university positions besides myself (held by Franz Lehner, Daniele D'Angeli and Ecaterina Sava-Huss); except for Lehner's position, no tenure is possible. Since 2002, I have run three subsequent FWF projects, each one with the funding corresponding to one PostDoc and one PhD position. These two projects have been the backbone for building up a vibrant research group of 9-12 people on the whole. Let me exemplify my way of operating. Agelos Georgakopoulos joined my group within the previous FWF project P19115-N18 and contributed significantly to the present project. Somewhat later he left, and now he is an associate professor at Warwick University, holding an ERC starting grant. He was not "lost" for the research group, because the exchange continued. At the same time, I could attract other young people to come to Graz, in particular Daniele D'Angeli, who integrated himself so well in the group that some time later he got a regular (though unfortunately not tenured) assistant professor position and recently obtained his "Habilitation". Similarly, the last project PostDoc Wojciech Cygan initially came only for 11 months, and then I managed to obtain a 6 months' funding from a different source. In the meantime, I was able to ensure that my project PhD Judith Kloas could be paid for her 4th year from a different source, so that I could re-employ Cygan within the project until July 31, 2017. This has been fruitful for both of them, because in this way they could extend their ongoing collaboration.

Thus, with careful eyes on different funding possibilities, occasionally shifting collaborators from one type of funding to another one, I could enlarge my group and offer research opportunities to twice as many persons as provided directly by the FWF funding. But I could not have done this without the FWF project. Concerning the scientific work itself, rather than primarily aiming at many joint papers of myself with the project collaborators, an important goal has always been to provide the collaborators with research problems and an environment in which they can sharpen their autonomy as well as widen their panorama. This comes of course along with a positive impact on my own work.

In concluding this initial statement, I can say that this as well as the preceding FWF projects have had an extremely positive impact on building up my research group, quite far beyond the formal scope of the project. I would have hoped that with another follow-up project, I might be able to support a good number of young persons in the years until my retirement scheduled for September 30, 2022.

1.2 Information on the development of the research project

The project proposal is still available under

<https://www.math.tugraz.at/~woess/FWF-P24028-proposal.pdf>

It owes much to my scientific exchange with **Agelos Georgakopoulos**. Based on this exchange, the proposed themes were

A. Continuation of the study of Brownian motion and harmonic functions on “treebolic space” and related horocyclic products.

B. Random walks on Baumslag-Solitar groups - related with A.

C. Hyperbolic extensions, stochastic dynamical systems, and processes on hyperbolic boundaries.

D. A continuation of the work on Brownian motion and potential theory on metric graphs with finite total edge lengths.

E. Characterisation of the planar groups, yielding structure tree splittings of these groups; extensions to non-planar cases.

F. Using augmented trees and Gromov-hyperbolicity to describe local connectedness in topological spaces.

At the end, Georgakopoulos was funded by the project only for 5 months, but he is an outstanding mathematician with many sparkling ideas, and undertook work in several of those topics, in particular D and E, as well as further ones. Thus, he contributed a good number of substantial papers which acknowledge support by this project. Topic A was primarily my own work. Comments on the other topics will follow below.

After the departure of Georgakopoulos (now on a permanent position at Warwick Univ.), the project was interrupted for 8 months in order to permit a thorough search for a new post doc. At a one week visit to the institute in autumn 2012, Daniele D'Angeli made a very good impression on all of us, and I offered the position to him. After PhD in Rome, he had gathered multiple international experience and started autonomous research with main focus in a field which is well related with the overall theme of the present project, namely

G. Groups acting on rooted trees, in particular automata groups.

D'Angeli started on April 15, 2013 and integrated himself perfectly in the research group and the institute, learned German within 8 weeks and turned out to be an optimal “acquisition”. Therefore, when a 6 years' regular assistant prof. position became vacant, he was ranked first on the list of applicants. He is now holding this position since March 1, 2014, so that his funding by the present project ended with that date. Due to his very fruitful research activities, he has contributed a large number of publications with support by the present project.

Also on March 1, 2014, Judith Kloas became project PhD. She has been working in the realm of topic C, namely on multidimensional reflected random walk. She was funded by the

present project until February 28, 2017. The “appearance“ of a different funding possibility made it seem wise to shift her payment to this other source starting with March 1, 2017.

From May 1 to December 31, 2014, my former PhD student **Wilfried Huss** held the PostDoc position within the project. He collaborated with Daniele D'Angeli, as well as with his wife and with Sebastian Müller on another issue that is well related with the “hyperbolic“ aspects of this project, namely

H. Rotor router walks on trees.

Coming back to the original topics, **B** was covered by **Johannes Cuno**, first as a PhD student in the FWF-Doctoral Program (DK) “Discrete Mathematics“, whose speaker I am. Subsequently, in the period May 15 – October 15, 2015, he concluded his PhD within the present project, where he remained also as a PostDoc until April 14, 2016 before going abroad.

The last project PostDoc was **Wojciech Cygan** (former PhD student of A. Bendikov, Wroclaw) from October 1, 2015 until July 31, 2017 with an interruption of 6 months (September 1, 2016 – February 28, 2017) during which he was funded by NAWI Graz. His contributions to the project were to part **C** in two ways: stochastic processes on ultrametric spaces (= boundaries of trees) and collaboration with Judith Kloas on variants of multidimensional reflected random walk and resulting issues of more probabilistic-analytic nature which point at new research to become part of a follow-up project.

In conclusion, there are project results and publications to 5 of the 6 initially outlined topics **A – E** (the only exception being **F**) plus the additional, related topics **G – H**.

1.3 Most important results and brief description of their significance

The rather long list of project publications is attached at the end and available at <https://www.math.tugraz.at/~woess/FWF-P24028publ.html>

Here, I highlight some selected publications, involving all project collaborators.

A. Georgakopoulos: "The boundary of a square tiling of a graph coincides with the Poisson boundary", *Inventiones Math.* 203 (2016) 773-821.

[DOI: 10.1007/s00222-015-0601-0](https://doi.org/10.1007/s00222-015-0601-0), [arXiv:1301.1506](https://arxiv.org/abs/1301.1506)

This is one of the flagships of Georgakopoulos' work in the last years. In ingenious work in the 1990ies, Benamou and Schramm [*Ann. Probab.* 24 (1996) 1219-1238] introduced the use of square tilings for the study of random walks and harmonic functions on planar graphs. In the transient case, the square tiling has a natural geometric boundary, and they asked whether this was the Poisson boundary. Now, the available toolbox for determining the Poisson boundary of random walks is mainly restricted to groups and generalisations thereof, see the profound work of Kaimanovich. In the present work, Georgakopoulos develops entirely new methods which apply to this setting which is in general far from groups. He uses what he calls sharp harmonic functions (basically corresponding to tail events). This is combined with a detailed description of the interplay between the “electrical“ properties of the random walk and the boundary of the square tiling in the cylinder on which the latter takes place. Referring to the project theme, the “hyperbolic“ significance of this work is underlined

by the recent paper by Federici and Georgakopoulos ["Hyperbolicity vs. amenability for planar graphs", *Discrete Appl. Geometry*, in print, [arXiv:1509.02028](https://arxiv.org/abs/1509.02028)].

D. D'Angeli and E. Rodaro: "Freeness of automata groups vs boundary dynamics", *Journal of Algebra* 462 (2016) 115–136. [arXiv:1410.6097](https://arxiv.org/abs/1410.6097)

Automata groups are a rich object of study from several viewpoints. They gained high significance through the work of Grigorchuk on groups with intermediate growth. Their exploration in terms of Mealy automata generating groups that act on a rooted tree owes much to the work of Sidki. One of the interesting directions is to study the relation between algebraic properties of the group and the dynamics of the action on the boundary. The authors undertake such a study in terms of the Schreier graphs associated with orbits of boundary points. Considering in particular automata which are bireversible transducers, they describe (non-)freeness of the resulting groups.

A. Bendikov, L. Saloff-Coste, M. Salvatori and W. Woess: "Brownian motion on treebolic space: positive harmonic functions", *Ann. Institut Fourier (Grenoble)* 66 (2016) 1691-1731. DOI: [10.5802/aif.3048](https://doi.org/10.5802/aif.3048), [arXiv:1412.2218](https://arxiv.org/abs/1412.2218)

This is the final part of a long and rather hard specific research topic, which links discrete and non-discrete objects. The focus is on the study of Laplace operators on spaces which have singularities at certain bifurcation lines, and the description of the resulting stochastic process as well as the positive harmonic functions. The concrete object here is the horocyclic product of a tree and hyperbolic upper half plane. Previous work with Brofferio and Salvatori concerned the analogous horocyclic products of two trees – purely discrete – and two hyperbolic planes – smooth. In the present situation, considerable difficulties had to be overcome, starting with the rigorous construction of the Laplacians, which was settled in a first, very general paper [*Advances in Math.* 226 (2011) 992-1055]. The evolution in space of the resulting Brownian motion on treebolic space was described in paper [5] of the publication list included below. The main challenges of the present paper concern the Poisson representation of harmonic functions on rectangular sets, its combination with suitable Harnack inequalities, and the proof that positive harmonic functions for the Laplacian arise as extensions of positive harmonic functions for a related random walk on the isometry group of the space. The main results concern a complete description of all positive harmonic functions.

J. Cuno and E. Sava-Huss: "Random walks on Baumslag-Solitar groups", *Israel J. Math.*, in print. [arXiv:1510.00833](https://arxiv.org/abs/1510.00833)

The treebolic spaces mentioned above are, with suitable parameters, spaces on which the amenable Baumslag-Solitar groups act by isometries and with compact quotient. This leads to a good understanding of the Poisson boundary of random walks on those groups, although this geometric viewpoint was not used in the corresponding paper by Kaimanovich [in "Probability measures on groups X " (Oberwolfach, 1990), Plenum, New York, 1991, pp. 205–238]. Now, as the present paper explains, also every nonamenable Baumslag-Solitar group acts co-compactly by isometries on a strip complex which arises as a level product of hyperbolic plane with a tree, where however the horocyclic level has to be replaced by a different level function. In this case, basically due to non-amenability, the geometric description of the Poisson boundary is harder than in the amenable situation. In this work, the main difficulty concerns the situation where the "vertical drift" is 0, which is tackled by smart methods.

A. Bendikov, W. Cygan and W. Woess: "Oscillating heat kernels on ultrametric spaces", J. Spectral Theory, in print. [arXiv:1610.03292](https://arxiv.org/abs/1610.03292)

This is one of the follow-ups of the long paper on isotropic processes on ultrametric spaces by Bendikov, Grigor'yan, Pittet and Woess [Russian Math. Surveys 69 (2014) 589-680], number [8] in the below list of project publications. Processes on such spaces have been studied by various authors. The clever and simple approach going back mainly to Bendikov allows an efficient study of many different features. The spectrum is completely understood, and a transparent presentation of the transition kernel leads to good asymptotic estimates. Here, we are concerned in particular with two group-invariant random processes of this type. The first is the heat kernel of the operator of fractional derivative of order α on the p -adic numbers. The precise asymptotics consist of a leading term $t^{-1/\alpha}$ multiplied by a non-constant *log*-periodic function. The second concerns a similar operator on the infinite symmetric group. The transition probabilities of the associated continuous time random walk oscillate in a more dramatic way between an upper and a lower function whose quotient tends to 0. The results are somehow reminiscent of similar oscillatory behaviour of random walks on the discrete Sierpinski gasket found by Grabner and Woess [Stochastic Process. Appl. 69 (1997) 127–138].

J. Kloas and W. Woess: "Multidimensional random walk with reflections", preprint. [arXiv:1707.01714](https://arxiv.org/abs/1707.01714)

Reflected random walk on the real or integer line is obtained by taking an ordinary random walk, but reflecting it (i.e., changing sign) whenever it becomes negative, before adding the next increment. It has been studied by various authors since the 1940ies, including Feller and Knight. It was considered within the context of *locally contractive* stochastic dynamical systems in an excellent PhD thesis by Benda [Ludwig-Maximilians-Universität München (1998)] and subsequently by Peigné and Woess [Colloq. Math. 125 (2011) 31-54]. The present paper concerns the multidimensional situation, where one has reflection in some or all coordinates, while there may be coordinates where the respective marginal random walk evolves without reflections. Some results in the purely non-lattice case are due to Peigné [Ann. Inst. H. Poincaré Probab. Stat. 28 (1992) 63–94]. In the PhD work of J. Kloas, it turned out that the multidimensional case is substantially harder than the onedimensional one. Even positive recurrence is a challenge, in particular in presence of marginals which are discrete (lattice). The present paper provides a detailed result on this question, when there is reflection in all coordinates, by using contractivity and convergence of the backward process (i.e., while the more direct approach of first determining explicitly the essential classes appears to comprise significant obstacles. Another result concerns the case when there are non-reflecting coordinates which are non-lattice: here, the methods use tools from ergodic theory on stationary random walks.

1.4 Information on the execution of the project, use of available funds and (where appropriate) any changes to the original project plan

- Duration: May 15 – September 14, 2012 and April 15, 2013 – November 14, 2017
- Use of personnel:
 - Agelos Georgakopoulos, PostDoc, May 15 – September 14, 2012
 - Daniele D'Angeli, PostDoc, April 15, 2013 – February 28, 2014
 - Judith Kloas, PhD, March 1, 2014 – February 28, 2017 and September 1 – November 14, 2017
 - Wilfried Huss, PostDoc, May 1 – December 31, 2014

- Johannes Cuno, PhD + PostDoc, May 15, 2015 – April 14, 2016
- Wojciech Cygan, PostDoc, October 1, 2015 – August 31, 2016 and March 1 – July 31, 2017
- Major items of equipment purchased: none
- Other significant changes: the additional topics G (groups acting on rooted trees, in particular automata groups) and H (Rotor router walks on trees), fitting well into the overall themes.

2. Personnel development – importance of the project for the research careers of those involved

- Agelos Georgakopoulos: September 17, 2012 – February 28, 2013 Zeeman Lecturer, Warwick Univ.; March 1, 2013 – June 30, 2015 assistant prof. and since July 1, 2015 associate prof. at Warwick Univ. Since September 2013 ERC starting grant.
- Daniele D’Angeli: currently on a regular 6 years’ assistant prof. position at the same institute.
- Judith Kloas: temporarily & after November 14, 2017 funded from a different source, PhD defense on March 2, 2018.
- Wilfried Huss: January 1, 2015 – March 31, 2016: FWF Schrödinger fellowship at Cornell University, USA. Subsequently return fellowship, senior PostDoc at Institut für Diskrete Mathematik, January 1, 2017 – February 28, 2018 assistant prof (maternity coverage of his wife’s position). Subsequently paternity leave.
- Johannes Cuno: October 15, 2015 – May 31, 2015: PostDoc at Institut für Diskrete Mathematik, part-time coordinator of the FWF-funded doctoral program “Discrete Mathematics” (of which I am the speaker). June 1, 2016 – July 31, 2017: PostDoc at Univ. Ottawa, Canada (Prof. Kaimanovich), subsequently: PostDoc at ENS, Paris (Prof. Erschler).
- Wojciech Cygan: regular assistant prof. position at Univ. Wroclaw, Poland. Since autumn 2017 PostDoc at Univ. Dresden, Germany (Prof. Schilling)

As one can see, for all of them, working within this project work has contributed significantly to their subsequent opportunities and research careers.

3. Further important aspects

3.1 Project-related participation in conferences and seminar talks

Wolfgang Woess (selection)

- 14.9.2012 Seminaire de Probabilités, Univ. Tours, FR (seminar talk)
- 5.-11.5.2013 workshop “Heat Kernels, Stochastic Processes and Functional Inequalities“, Oberwolfach, DE (talk)
- 24.-25.9.2013 conference "First Austrian Stochastic Days", Linz, AT (talk)

- 4.-7.12.2013 conference “Mathematical Technology of Networks”, Bielefeld, DE (invited talk)
- 24.3.2014 Séminaire de Probabilités et Théorie Ergodique, Univ. Tours, FR (seminar talk)
- 18.-22.5.2015 conference “Random walks on graphs and potential theory“, Univ. Warwick, UK (invited talk)
- 1.-5.6.2015 International Conference on Boundaries and Ergodic Geometry, Univ. Notre Dame, US (invited talk)
- 29.-30.10.2015 “2nd Zurich / Strasbourg - Meeting: Frontiers in Analysis and Probability”, Zürich, CH (invited talk)
- 24.-21.5.2016 conference “Probabilistic Aspects of Harmonic Analysis“, Bedlewo, PL (invited talk)
- 30.5.-3.6.2016 conference “XXXVI Convegno di Analisi Armonica“, Gargnano, IT (invited talk)
- 18.-21.7.2016 conference “Stochastic processes under constraints“, Augsburg, DE (invited talk)
- 20.-22.6.2016 Final conference of the MADACA project, Domaine de Chalès, FR (invited talk)
- 22.-24.2. 2017 conference “Groups, Probability, Dynamics; Rome, IT (organiser)
- 31.7.-4.8.2017 conference “Analysis and Geometry on Graphs and Manifolds“, Potsdam, DE (invited talk)
- 21.11.-21.12.2017 conference “Discrete and Continuous Models in the Theory of Networks“, Bielefeld, DE (invited talk)

Agelos Georgakopoulos:

- 24.-26.5.2013 26th Cumberland Conference on Combinatorics, Graph Theory & Computing, UK (invited talk)
- 9.-13.9.2013 European Conference on Combinatorics, Graph Theory and Applications, Pisa, IT (talk)
- 16.9.2013 seminar talk, Group Theory Seminar, Univ. Rome III, IT.
- 7.10.2013 seminar talk, Combinatorics Seminar, Cambridge Univ. , UK
- 27.-31.1.2014 conference “Random Walks on Groups“, Paris, FR (talk)
- 2.-6.6.2014 conference “Groups, Graphs, and Random Walks“, Cortona, IT (talk)
- 14.-19.9.2014 workshop ”Probability on Trees and Planar Graphs“, Banff, CA (invited talk)

Daniele D’Angeli

- 25.4.2013 seminar talk, Structure Theory Seminar, TU Graz, AT
- 2.7.2013 Workshop "Groups, graphs, random processes", TU Graz (talk)
- 23.-27.9.2013 ÖMG – DMV congress, Innsbruck, AT (talk)
- 2.-6.6.2014 conference “Groups, Graphs, and Random Walks“, Cortona, IT (talk)
- 15.7.2014 seminar talk, Univ. Porto, PT
- 16.12.2014 seminar talk, Univ. Marseille, FR
- 10.-13.6.2015 special session “New Trends in Automata Groups and Semigroups” (organiser), AMS-EMS-SPM International Meeting 2015, Porto, PT
- 29.3.-2.4.2016 conference “Ischia Group Theory“ (talk)
- 22.-24.2. 2017 conference “Groups, Probability, Dynamics; Rome, IT (organiser)

Judith Kloas

- 2.-2.-6.6.2014 conference “Groups, Graphs, and Random Walks”, Cortona, IT
- 5.-9.1.2015 winter school “Random walks and Dunkl processes”, Tours, FR
- 18.-22.5.2015 conference “Random walks on graphs and potential theory”, Univ. Warwick, UK
- 28.-29.09.2015 conference "Austrian Stochastic Days", Vienna, AT (talk)
- 22.-23.10.2015 meeting “Young female researchers in probability”, Berlin (poster)
- 21.-25.09.2015 seminar talk & research stay at Univ. Aix-Marseille
- 4.-9.1.2016 seminar talk & research stay at Univ. Tours
- 1.-4.3.2016 German Probability and Statistics Days, Bochum (talk)
- 27.-31.3.2017 Spring school in mathematics, TU Darmstadt (talk)
- 11.-13.9.2017 ÖMG - DMV Congress 2017 / "Austrian Stochastic Days", Salzburg, AT (talk)

Wilfried Huss

- 2.-6.6.2014 conference “Groups, Graphs, and Random Walks”, Cortona, IT
- 28.-25.10.2016 seminar talk & research stay at Univ. Aix-Marseille

Johannes Cuno

- 7.-11.12.2015 conference “Random Walks and Harmonic Functions on Groups”, Lausanne, CH (talk)
- 24.11.2015 seminar talk, Geometry and Analysis on Groups, Univ. Vienna, AT
- 2.11.2016 seminar talk, Analysis Seminar University of Ottawa, CA
- 25.9.2017 Réunion de rentrée de l'équipe «Algèbre et géométrie» ENS, Paris, FR (talk)

Wojciech Cygan

- 24.-21.5.2016 conference “Probabilistic Aspects of Harmonic Analysis”, Bedlewo, PL (italk)
- 30.6-01.7.2016 conference "Austrian Stochastic Days", Graz, AT (talk)
- 20.-22.6.2016 Final conference of the MADACA project, Domaine de Chalès, FR
- 25-29.7.2016 8th International Conference on Lévy Processes, Angers, FR
- 15-19.5.2017 conference Probability and Analysis, Bedlewo (Poland)
- 24-28.7.2017 The 39th Conference on Stochastic Processes and their Applications (SPA2017), Moscow, RU
- 4-8.9.2017 “Analysis and Applications”. A conference in honor of Elias M. Stein, Wrocław, PL

3.2 Organisation of symposia and conferences

- 9.-11.6.2012 Focussed meeting: “the shape of branching random walk and related topics”, Graz, AT (Organisers: E. Sava-Huss and W. Woess)
- 15.-18.9.2013 Leoben-Ljubljana Graph Theory Seminar 2013, Graz, AT (Organisers: W. Huss and W. Woess)
- 30.6.-1.7.2016 Austrian Stochastics Days, Graz, AT (Organiser: W. Woess)

- 25.28.9.2016 Leoben-Ljubljana Graph Theory Seminar 2016, Judenburg, AT (Organisers: J. Cuno, I. Peterin and W. Woess)
- 22.-24.2.2017 conference “Groups, Probability, Dynamics“, Rome, IT (Organisers: A. Donno, F. Scarabotti, F. Tolli , D. D'Angeli, W. Woess)

3.3 Funding principles

My primary goal has always been to use the project funding for supporting young scientists, that is, for their salaries. As mentioned above, optimising these possibilities resulted sometimes in “shifting” the collaborators to different funding sources. For example, Cygan could be paid by a PostDoc funding of NAWI Graz (the cooperation between the two Graz universities) for an interim period of 6 months. Without this, he would have left already after August, 2016, but then I was able to keep him here until July, 2017: also thanks to the fact that Judith Kloas could be “shifted” to another PhD funding source, additional funding became available for Cygan beginning with March 2017. This was advantageous for both of them, since they had started a fruitful collaboration. For the same reason (maximising the funding for young scientists), I hardly ever use project money for my own travel to conferences, etc. Needless to say that nevertheless, on the non-financial level, I profited enormously from the project.

3.4 International collaboration partners

Several names have been listed in the project application, including possibly funding their visits by the project. Of those listed, Bendikov has been a frequent guest as visiting professor and for shorter visits, Lenz came for a one-month visiting professorship, and Kaimanovich also came as a visiting professor and as one of the referees of Cuno’s PhD defense. Mohar could only come for a very short visit, and Saloff-Coste was too busy as the chairman at Cornell to detach himself for coming to Graz. (Both Kaimanovich and Saloff-Coste have been hosting project group members as guests or PostDocs.) In any case, it was almost always possible to fund these guests from other sources, so that my aim to use the project funding primarily for the salaries of young project collaborators has remained intact throughout all years.

3.5 Doctoral Program “Discrete Mathematics”

Since 2010 we are running a big doctoral program (“Doktoratskolleg” = DK) with substantial funding from the FWF. It unites colleagues from TU Graz and the universities of Graz and Leoben in a kind of “excellence program” of joint doctoral education. Our interpretation of “Discrete Mathematics” is a wide one, going well beyond combinatorics, graph theory and computer science related topics. Our range includes number theory, algebra, discrete geometry, fractals & dynamics, combinatorial optimisation, random graphs and random processes. See <https://www.math.tugraz.at/discrete/> for all details. I am the speaker (= chief responsible) for this program. It had of course an outreach on the present project, regarding the enhanced scientific environment in general, and the fact that the project PhD students were attached to that program: Cuno first as an internal DK student (funded completely by that source) and in the final phase as an associated DK student (funded by the present project, but taking advantage of the DK environment and activities), and Kloas as an associated DK student. Also, after graduation Cuno was for more than half year part time coordinator of the DK. The interplay between the DK and several other smaller research projects has been crucial for all involved sides, including the present project.

III. Attachments

1. Scholarly / scientific publications

At this point, the FWF guidelines say “Publications may only be listed if they relate directly to the project”. The number of publications acknowledging support from this project is quite high. But in view of my approach how to run such a project, it is hard to distinguish which publications should or should not be directly related to the project. Of course, for example the first paper of Georgakopoulos in the list (with Benjamini and Curien) was begun before the project started, but concluded in Graz and is clearly part of the project substance. His Inventiones paper was begun here, and terminated later. Cygan, while being funded by this project, collaborated not only with myself, Bendikov, and Kloas, but also with other colleagues from Wroclaw, and there neither was a good reason to block this work nor is there a good reason why it should not be considered being related with the project. Similar arguments apply to the work of D’Angeli, where indeed it was primarily his employment in this project which opened for him the opportunities for very high research activities resulting in an impressive number of good publications. And finally, as I explained above, I took hardly any project money for my own travel purposes or similar – nevertheless, my own listed publications are in my understanding project publications in every respect. In view of these considerations, below I’m listing *all* publications which acknowledge project support.

Open Access: with very few exceptions, which are indicated explicitly, all publications have Green Open Access via posting on arXiv.

1.1 Peer-reviewed publications / already published

All papers are openly accessible via at least one of DOI, article link, or arXiv

[1] I. Benjamini, N. Curien, A. Georgakopoulos: "The Liouville and the intersection properties are equivalent for planar graphs", Electron. Commun. Probab. 17 (2012), paper no. 42, 5pp. **Gold OA**, <https://projecteuclid.org/euclid.ecp/1465263175> and [arXiv:1203.4002](https://arxiv.org/abs/1203.4002)

[2] G. Chacon, R. Colucci and D. D’Angeli: "Density of backward branches on the Julia set of a semigroup", Sarajevo Journal of Mathematics 10 (2014) 77-87. **Gold OA**, [DOI: 10.5644/SJM.10.1.10](https://doi.org/10.5644/SJM.10.1.10)

[3] G. Chacon, R. Colucci and D. D’Angeli: "Recurrence analysis on Julia sets of semigroups of complex polynomials", Journal of Applied Mathematics and Computing 46 (2014) 201-214. [DOI: 10.1007/s12190-013-0746-1](https://doi.org/10.1007/s12190-013-0746-1), [download link](#)

[4] D. D’Angeli: "Schreier graphs of an extended version of the binary adding machine", Electronic Journal of Combinatorics 21 (2014), no. 4, paper 4.20. **Gold OA**, [Link to journal article](#).

[5] A. Bendikov, L. Saloff-Coste, M. Salvatori and W. Woess: "Brownian motion on treebolic space: escape to infinity", Revista Matemática Iberoamericana 31 (2015) 935–976. [DOI: 10.4171/RMI/859](https://doi.org/10.4171/RMI/859), [arXiv:1212.6151](https://arxiv.org/abs/1212.6151)

[6] A. Georgakopoulos: "The boundary of a square tiling of a graph coincides with the Poisson boundary", Inventiones Math. 203 (2016) 773-821. [DOI: 10.1007/s00222-015-0601-0](https://doi.org/10.1007/s00222-015-0601-0), [arXiv:1301.1506](https://arxiv.org/abs/1301.1506)

- [7] A. Georgakopoulos, S. Wagner: "Hitting times, cover cost, and the Wiener index of a tree", *Journal of Graph Theory* 84 (2017) 311–326. [DOI: 10.1002/jgt.22029](https://doi.org/10.1002/jgt.22029), [arXiv:1302.3212](https://arxiv.org/abs/1302.3212)
- [8] A. Bendikov, A. Grigor'yan, Ch. Pittet and W. Woess: "Isotropic Markov semigroups on ultra-metric spaces", *Uspekhi Mat. Nauk* 69 (2014) No. 4 (418), 3-102 (Russian version). English original in *Russian Math. Surveys* 69 (2014) No. 4, 589-680. [DOI: 10.1070/RM2014v069n04ABEH004907](https://doi.org/10.1070/RM2014v069n04ABEH004907), [arXiv:1304.6271](https://arxiv.org/abs/1304.6271)
- [9] J. Parkinson and W. Woess: "Regular sequences and random walks in affine buildings", *Ann. Institut Fourier (Grenoble)* 65 (2015) 675-707. [DOI: 10.5802/aif.2941](https://doi.org/10.5802/aif.2941), [arXiv:1308.5120](https://arxiv.org/abs/1308.5120)
- [10] A. Georgakopoulos, S. Haeseler, M. Keller, D. Lenz, R. K. Wojciechowski: "Graphs of finite measure", *J. Math. Pures Appl.* 103 (2015) 1093–1131. [DOI: 10.1016/j.matpur.2014.10.006](https://doi.org/10.1016/j.matpur.2014.10.006), [arXiv:1309.3501](https://arxiv.org/abs/1309.3501)
- [11] D. D'Angeli and E. Rodaro: "Groups and semigroups sdefined by colorings of synchronizing automata", *Internat.l Journal of Algebra and Computation* 46 (2014) 201-214. [DOI: 10.1142/S0218196714500337](https://doi.org/10.1142/S0218196714500337), [arXiv:1310.5242](https://arxiv.org/abs/1310.5242)
- [12] D. D'Angeli and E. Rodaro: "A geometric approach to (semi)groups defined by automata via dual transducers", *Geometriae Dedicata* 174 (2015) 375-400. [DOI: 10.1007/s10711-014-0024-x](https://doi.org/10.1007/s10711-014-0024-x), [arXiv:1403.1722](https://arxiv.org/abs/1403.1722)
- [13] D. D'Angeli and A. Donno: "Isomorphism classification of infinite Sierpinski carpet graph", in: *Proceedings of the First Minisymposium on Mathematics in Engineering and Technology, ICNAAM 2014. AIP Conference Proceedings* 1648, 570002 (2015). [DOI: 10.1063/1.4912788](https://doi.org/10.1063/1.4912788), [arXiv:1802.09839](https://arxiv.org/abs/1802.09839)
- [14] D. D'Angeli, E. Sava-Huss and A. Donno: "Connectedness and isomorphism problems of the zig-zag product of graphs", *Journal of Graph Theory* 83 (2016), 120-151. [DOI: 10.1002/jgt.21917](https://doi.org/10.1002/jgt.21917), [arXiv:1404.4342](https://arxiv.org/abs/1404.4342)
- [15] D. D'Angeli and E. Rodaro: "Freeness of automata groups vs boundary dynamics", *Journal of Algebra* 462 (2016) 115–136. [DOI: 10.1016/j.jalgebra.2016.05.015](https://doi.org/10.1016/j.jalgebra.2016.05.015), [arXiv:1410.6097](https://arxiv.org/abs/1410.6097)
- [16] A. Bendikov, L. Saloff-Coste, M. Salvatori and W. Woess: "Brownian motion on treebolic space: positive harmonic functions", *Ann. Institut Fourier (Grenoble)* 66 (2016) 1691-1731. [DOI: 10.5802/aif.3048](https://doi.org/10.5802/aif.3048), [arXiv:1412.2218](https://arxiv.org/abs/1412.2218)
- [17] W. Huss, S. Müller, E. Sava-Huss: "Rotor-routing on Galton-Watson trees", *Electronic Communications in Probability* 20 (2015); 1-12. **Gold OA**, [DOI: 10.1214/ECP.v20-4000](https://doi.org/10.1214/ECP.v20-4000), [arXiv:1412.5330](https://arxiv.org/abs/1412.5330)
- [18] D. D'Angeli and A. Donno, "Metric compactification of infinite Sierpinski carpet graphs", *Discrete Math.* 339 (2016), 2693-2705. [DOI: 10.1016/j.disc.2016.04.023](https://doi.org/10.1016/j.disc.2016.04.023), [arXiv:1501.03178](https://arxiv.org/abs/1501.03178)
- [19] I. Bondarenko, D. D'Angeli and E. Rodaro: "The lamplighter group $Z_3 \wr Z$ generated by a bireversible automaton", *Communications in Algebra Comm. Algebra* 44 (2016) 5257-5268. [DOI: 10.1080/00927872.2016.1172602](https://doi.org/10.1080/00927872.2016.1172602), [arXiv:1502.07981](https://arxiv.org/abs/1502.07981)

- [20] D. D'Angeli, A. Donno and A. Monti: "Computing the Wiener index in finite Sierpinski carpet graphs", AIP Conference Proceedings, ICNAAM 2015, Rhodes. AIP Conference Proceedings 1738, 270008 (2016). [DOI: 10.1063/1.4952047](https://doi.org/10.1063/1.4952047), [arXiv:1802.09840](https://arxiv.org/abs/1802.09840)
- [21] A. Bendikov, W. Cygan and B. Trojan: "Limit theorems for random walks", Stochastic Processes Appl. 127 (2017) 3268–3290. [DOI: 10.1016/j.spa.2017.02.008](https://doi.org/10.1016/j.spa.2017.02.008), [arXiv:1504.01759](https://arxiv.org/abs/1504.01759)
- [22] W. Cygan and T. Grzywny: "Heat content for convolution semigroups", J. Math. Anal. Appl. 446 (2017) 1393-1414. [DOI: 10.1016/j.jmaa.2016.09.051](https://doi.org/10.1016/j.jmaa.2016.09.051), [arXiv:1606.09168](https://arxiv.org/abs/1606.09168)
- [23] D. D'Angeli: "Horofunctions on Sierpinski type triangles", Utilitas Mathematica 17 (2017) 267-277. [arXiv:1707.06030](https://arxiv.org/abs/1707.06030)
- [24] I. Bondarenko, D. D'Angeli and T. Nagnibeda: "Ends of Schreier graphs and cut-points of limit spaces of self-similar groups", Journal of Fractal Geometry 4 (2017) 369–424. [DOI: 10.4171/JFG/55](https://doi.org/10.4171/JFG/55), [arXiv:1601.07587](https://arxiv.org/abs/1601.07587)

1.2 Peer-reviewed publications / accepted for publication

- [25] J. Cuno and E. Sava-Huss: "Random walks on Baumslag-Solitar groups", Israel J. Math., in print. [arXiv:1510.00833](https://arxiv.org/abs/1510.00833)
- [26] A. Bendikov, W. Cygan and W. Woess: "Oscillating heat kernels on ultrametric spaces", J. Spectral Theory, in print. [arXiv:1610.03292](https://arxiv.org/abs/1610.03292)
- [27] W. Cygan and J. Kloas: "On recurrence of the multidimensional Lindley process", Electronic Communications in Probability, to appear . [arXiv:1707.01714](https://arxiv.org/abs/1707.01714)
- [28] J. Kloas and W. Woess: "Multidimensional random walk with reflections", Stochastic Processes Appl., to appear. [arXiv:1704.06055](https://arxiv.org/abs/1704.06055)

1.3 Preprints submitted for publication

- [29] A. Georgakopoulos: "On walk-regular graphs and graphs with symmetric hitting times", preprint. [arXiv:1211.5689](https://arxiv.org/abs/1211.5689)
- [30] A. Georgakopoulos, K. Kolesko: "Brownian Motion on graph-like spaces", preprint. [arXiv:1405.6580](https://arxiv.org/abs/1405.6580)
- [31] W. Cygan and T. Grzywny: "A note on the generalized heat content for Lévy processes", preprint. [arXiv:1703.10790](https://arxiv.org/abs/1703.10790)

1.4 PhD theses

- [A] J. Cuno: "Combinatorial and Probabilistic Aspects of Discrete Groups", TU Graz, 2015.
- [B] J. Kloas: "Reflected and stopped random walks, and the distinguishing number of graphs", TU Graz, 2018.

2. Development of collaborations

N	G	E	D	Name	Institution
RUS	M	E3	W	Alexander Bendikov	Univ. Wroclaw, POL
ITA	M	E3	W	Alfredo Donno	Univ. Cusano, Rome, ITA
AUS	M	E3	W	James Parkinson	Univ. Sydney, AUS
ITA	M	E3	W	Emmanuele Rodaro	Univ. Milano, ITA
USA	M	E2	W	Laurent Saloff-Coste	Cornell Univ., NY, USA

3. Development of human resources in the course of the project

	In progress	Completed	Gender	
			f	m
Full professorship	1			1
<i>Venia</i> thesis (<i>Habilitation</i>) / Equivalent senior scientist qualification		2		2
Postdoc	4	5		5
Ph.D. theses		2	1	1

Remarks: Georgakopoulos is shortlisted for promotion to Reader (equiv. Prof. !) in Warwick, resp. Prof. In Bath

Georgakopoulos obtained Habilitation in Hamburg, D'Angeli in Graz

Georgakopoulos has completed his PostDoc years.

D'Angeli, Huss, Cuno, Cygan have completed PostDoc positions and now have other PostDoc positions. Thus, the line "PostDoc" refers to 5 persons.

Cuno and Kloas have completed their PhD theses.

For details, see Section II.2.

4. Applications for follow-up projects

(FWF projects)

Project number (if applicable)	P31237-N35		
Project type	FWF stand alone project		
Title / subject	Walks and boundaries – a wide range		
Status	granted <input type="checkbox"/>	pending <input checked="" type="checkbox"/>	in preparation <input type="checkbox"/>
Application reference (if a patent is applied)			

(current status: rejected on the basis of 3 reviews with 10 "excellent" and 2 "very good", all stating "funding with highest priority". Still pending in view of possible "Matching Funds" agreement with the region of Styria.)

IV. Cooperation with the FWF

Scale: -2 highly unsatisfactory
+1 satisfactory

-1 unsatisfactory
+2 highly satisfactory

0 appropriate
X not used

Rules

	Rating
Application guidelines	
Length	0
Clarity	0
Intelligibility	0

Procedures (submission, review, decision)

	+1
Advising	
	+1
Duration of procedure	
	-1
Transparency	

Project support

Advising	Availability	+2
		+2
	Level of detail	
		+1
	Intelligibility	

		+2
Financial transactions		
(credit transfers, equipment purchases, personnel management)		

Reporting / review / exploitation

		0
	Effort	
		0
	Transparency	
		x
	Support in PR work / exploitation	

Comments on cooperation/interaction with the FWF:

A large, empty rectangular box with a thin black border, intended for the user to provide comments on cooperation or interaction with the FWF. The box is currently blank.