Random Walks on Random Partial Graphs of transitive Graphs

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Summary of the project **P18703** of the Austrian Science Fund (FWF), carried out within the period of October 2006 until November 2009 at the Institute of Mathematics C of Graz University of Technology.

Contents

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1 Goals of the Project

1.1 Random Walks on Random Subgraphs

The initial motivation of the project was given by the results of [1], in which an estimate of the expected return probability lead to a bound of the expected number of open clusters per vertex in percolation theory. The method ('Interlacing') used is an eigenvalue comparison between some initial finite graph, and a partial graph (i.e. subgraph with a reduced set of edges). In [2] this method was developed further to show that an improvement of the expected return probability of a delayed random walk could be obtained.

The interlacing-method uses that a *finite* delayed random walk is (i) monotone under removal of edges, and (ii) that, by the theory of finite rank perturbations of self-adjoint operators, the change under a controlled amount of edge-removals (in the graph) allows eigenvalue comparison between the graph Laplacians of initial and perturbed finite graph.

At the beginning of the project, four goals were set, namely:

I. to obtain **lower bounds** for the return probability of finite random walks, in a similar manner as those in [1],

II. to find upper and lower estimates of the expected return-probability to **critical** percolation clusters, or, more generally, to random subgraphs with cluster-sizes having distributions without integral moments,

III. to extend the interlacing methods to **infinite graphs**,

IV. to obtain estimates of **specific types of invariant measures** generating random subgraphs, in which typical configurations dominate and characterise the corresponding asymptotic type of the return probability.

1.2 Random Subgraphs and Amenability

Horocyclic products of trees are graphs of exponential growth, which nevertheless may be amenable. The Diestel-Leader graphs are horocyclic products of homogeneous trees and have been classified into the symmetric/amenable, and the unsymmetric/non-amenable ones. Since much research has been carried out in the group of Prof. Woess concerning Diestel-Leader graphs, the following goal had emerged at the beginning of the project:

V. A family of percolation graphs 'interpolating' between an amenable and a non-amenable Diestel-Leader graph would induce the question of the existence of a phase transition. The idea is that the one type of graph is obtained if all edges are removed, while the other type of graph emerges, if all edges are left being part of the graph. The question is when the transition occurs. Equivalently, the issue is that of the **stability of amenability** under random perturbations (such as 'Percolation').

2 Achieved Results

Altogether, eight publications have emanated during the time of the project's duration. One has been published, two of them have been accepted, three of them are under review, and two are about to be submitted. One of the accepted ones is a proceedings volume of a workshop ('The Alp-Workshop 2009'), that has been financed by the project's budget. The titles and abstracts are listed and briefly discussed.

2.1 Random Walks on Random Subgraphs

[3]: 'An interlacing technique for spectra of random walks and its application to finite percolation clusters'; accepted for publication by: Journal of Theoretical Probability

A comparison technique for finite random walks on finite graphs is introduced, using the well-known interlacing method. It yields improved return probability bounds. A key feature is the incorporation of parts of the spectrum of the transition matrix other than just the principal eigenvalue. As an application, an upper bound of the expected return probability of a random walk with symmetric transition probabilities is found. In this case, the state space is a random partial graph of a regular graph of bounded geometry and transitive automorphism group. The law of the random edge-set is assumed to be stationary with respect to some transitive subgroup of the automorphism group ('invariant percolation'). Given that this subgroup is unimodular, it is shown that stationarity strengthens the upper bound of the expected return probability, compared with standard bounds derived from the Cheeger inequality.

This is an improvement of the original version of the paper in various aspects: By using a bisection algorithm for a spanning tree of the finite graph, the comparison theorem for the eigenvalues has been improved, such that there is a general bound for the **j-th eigenvalue**, only depending on the highest degree, the index 'j', and the size of the vertex set (Theorem 1.11). The resulting bounds for the expected return probability of the delayed random walk on percolation subgraph of transitive graphs G improve the smaller the degree of G.

[4]: 'Bounds for the annealed return probability on large finite random percolation clusters'; submitted to 'Mathematische Zeitschrift'

By an eigenvalue comparison-technique polynomial bounds for the expected return probability of the delayed random walk on critical Bernoulli bond percolation clusters are derived. The results refer to invariant percolations on unimodular transitive planar graphs with almost surely finite critical clusters. Estimates for the integrated density of states of the graph Laplacian of the two-dimensional Euclidean lattice follow. The upper bound which also applies to non-planar graphs relies on the fact that Cartesian products of finite graphs with cycles of a certain minimal size are Hamiltonian. The lower bound involves an upper estimate of the isoperimetric number ('Cheeger-constant') of finite graphs.

(This concerns Projects I. and II.) The upper bound is due to an interlacing technique involving **Hamiltonian graphs**. The lower bounds require the assumption of planarity. Communication with Prof. G. Grimmett (correspondence attached in this report) has indicated that the given polynomial bounds for Bernoulli percolation on the homogeneous trees correctly differentiate between the situation of the finite clusters and the incipient infinite cluster. This is one of the main statements of the paper.

[5]: 'Bounds for the annealed return probability on the incipient infinite cluster'

(Concerns Project III.) This refers to a chapter of a former version of [4]. It has been removed from the current version from that paper as it has became clear that a separate publication seemed appropriate (the method doesn't only apply to critical percolation, but to infinite graphs, in general).

In order to extend [4] to incipient clusters, upper and lower bounds of the size-distribution of balls in the incipient cluster are needed. Such bounds have been found by Kozma and Nachmias, recently, using an argument which they attribute to Barsky and Aizenmann. These bounds imply, by results of [5], that the well-known sub-diffusivity of the quenched return probability bounds of Kesten is not visible in the annealed bounds. In other words, taking the expected value of the return probability makes the sub-diffusivity vanquish.

2.2 Random Subgraphs and Amenability

For project V. three papers have been emerged. The general idea is to characterise amenability of horocyclic products of trees different than homogeneous ones. About the latter, it has been shown by Woess that amenability prevails if and only if the degree of the two trees involved are equal.

[6]: 'Amenability of horocyclic products of percolation trees'; submitted to Markov Processes and Related Fields

For horocyclic products of percolation subtrees of regular trees, we show almost sure amenability. Under a symmetry condition concerning the growth of the two percolation trees, we show the existence of an increasing Følner sequence (which we call **strong amenability)**.

The following results are obtained: Looking at the percolation trees as Galton-Watson trees their horocyclic product is amenable if and only if the convex hull of the offspring distributions have non-empty intersection; if the expected numbers of offspring of the two trees are equal, there is an exhausting Følner sequence. Moreover, this Følner sequence is explicitly identified.

[7] (with V. Kaimanovich) 'Stochastic homogenization of horospheric tree products'

We construct measures invariant with respect to equivalence relations which are graphed by horospheric products of trees. The construction is based on using conformal systems of boundary measures on treed equivalence relations. The existence of such an invariant measure allows us to establish amenability of horospheric products of random trees.

This is a generalisation of the results obtained in [6] to rooted trees which have the property that upper and lower exponential growth rate are equal. We call the associated tree-measures '**conformal** trees-measures', as they are invariant under shifting the root to one of its neighbours.

[8] (with Daniel Lenz and Ivan Veselic) '(Non-)Amenability of horocyclic products of uniformly growing trees', (preprint: http://users.minet.uni-jena.de/∼se64hod/uniform.pdf)

We characterise amenability of horocyclic products of uniformly growing trees with a non-vanishing exponential growth rate.

In this paper, a strong assumption on the uniformity over all centers of balls concerning the speed of convergence of their volume towards the asymptotic (exponential) behaviour leads to the possibility of also proving **non-amenability**. The method of using positive superharmonic functions was suggested by Prof. Wolfgang Woess.

3 Workshop/Applications

3.1 The Alp-Workshop 2009

With the budget of project P18703 available for special activities aside of personnel and travel, it was decided to organise a workshop in one of the pleasant small towns in the mountains of Graz' immediate surrounding. Originally planned as a small get-together of people working in the spectral theory of graphs, the prospect of locating the date of the event immediately after the 'Boundary workshop' made it possible to organise it as a satellite-meeting to this much bigger conference.

There were 12 distinguished speakers, and almost 40 participants. The stress was put on three main sub-topics: Discrete Probability (Random walks and random environments; T. Nagnibeda, T. Turova), Operator theory (Spectral theory of graphs; C. Pittet, I. Veselic), and Ergodic theory (Invariant measures on sets of rooted trees; V. Kaimanovich, J. Schmeling).

[9] The proceedings of this meeting will be published together with those of the 'Boundary workshop' in the series 'Progress in Probability' of Birkhäuser Publishing company.

3.2 Queueing Theory

[10] (with G.Rappitsch, E.Stadlober): 'Tandem Queues for Inventory Management under random perturbations', submitted by invitation to the journal Quality and Reliability Engineering International, special issue (2010)

Using the theory of M/M/1 queues at stationarity, we provide criteria of stability (recurrence) for a stochastic inventory model with an observed selling rate and optimally chosen buying rate. Optimality is based on the maximum gain under stability, where buying and selling-prices, as well as shop- and stock-keeping costs are incorporated into the model. An important aspect is to achieve robustness of the stocking process by minimizing the fluctuation of the predicted gain. This robustness can be achieved by controlling intermediate transfer rates of the assumed stochastic tandem network. Stochastic simulations demonstrate the applicability of the stability criteria under several scenarios of differing intensities of perturbation.

This unforeseen sideline of the project is a collaboration with Prof. Ernst Stadlober, from the Statistics-Dept. of TU-Graz, and Gerhard Rappitsch, from SensorDynamics AG, Austria. We developed a queueing model for an optimal stock-keeping strategy. The paper was introduced at the ENBIS 2009 (conference for statistics in industry). It was subsequently invited to be published in the journal Quality Management. It has been submitted and is currently still under review.

Our model involves tandem-queues and is strongly related to random walks on the quarter plane. While the question of recurrence has been settled for homogeneous random walks on the quarter plane (see the book by Malyshev, Menshikov, Fayolle), perturbations of the involved conductances still represent an open problem. Our results concern a special random perturbation of this model. In inventory science, our setup represents a so called 'two-echelon' system. Our results concerning the optimal strategy is an extension of the famous 'Newsboy problem' (http://en.wikipedia.org/wiki/Newsvendor).

4 Emerged Collaborations

4.1 Vadim Kaimanovich

During his visits to Graz, as well as during the time of the Seminar on Amenability at the Erwin Schroedinger Institute (http://www.esi.ac.at/further/KS2.pdf), it was possible to present and discuss the idea of [6] with Prof. Vadim Kaimanovich. He suggested to cooperate and approach the problem with the theory of graphed equivalence relations. This resulted in further visits to Vienna and to Bremen, and the completion of [**??**]. During his last visit to Jena on the 28th of April (http://www.analysis-lenz.uni-jena.de/Events.html), we worked on another joint paper, in which particularly the problem of finding examples for 'conformal trees' other than the so called 'Augmented Galton Watson trees'. This particularly concerns project IV., as if such a measure would be found, the results of [2, 3] on the expected return probability of random walks could be applied.

4.2 Tatyana Turova

During the alp-workshop, a discussion about the possibility to extend results concerning the integrated density of states of random graphs emerged. The talk presented by Peter Müller on Erdös-Renyi Random graphs suggested that for the so called Inhomogeneous Random Graph Model, which was the topic of the talk by Prof. Tatyana Turova at the workshop, the same type of statements might be obtained. Subsequently, an invitation by Prof. Turova to the University of Lund for one week in October of 2009 followed; during this time, the problem of showing so called Lifshitz-tails at the bottom of the spectrum of graph Laplacians of the Bolobás-Rjordan-Janson model was formulated, and the essential difficulties identified. The continuation and completion of this paper is planned to be carried out in the second half of this year.

4.3 Steven Lalley

Prof. Steven Lalley has been visiting TU-Graz as a collaborator of Prof. W. Woess, and for giving a course in discrete potential theory at Institute C of the Mathematics Department. During this time, one of the problems initially suggested in the proposal of the project (recurrence of random walks on growing balls of transient graphs) has been discussed with Prof. Lalley, who has suggested a solution for the corresponding problem of Brownian motion on balls in Euclidean space. The paper should be finished by the end of July.

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9. 'Boundaries and Spectra of Random Walks', Editors: D. Lenz, F. Sobieczky, W. Woess, Birkhäuser, Proceedings, Graz - St. Kathrein 2009; Progress in Probability, to appear in (2010)

10. F. Sobieczky, G. Rappitsch, E. Stadlober: 'Inventories modelled by stable tandem-queues under perturbation', submitted by invitation to QREI special issue (Quality and Reliability Engineering International), (2010)

6 List of most important Talks, Invitations

6.1 Invitations to Talks

1. 16.-24 May, 2007: 'Frontier Probability Days 2007', Colorado USA: During this conference, I could present my work concerning project V. to Prof. Yu Zhang of the University of Colorado Springs. He helped with several useful remarks, as to the completion of [6]. I will visit him during this summer and give a presentation at the local Seminar. As he is an expert in percolation theory, I hope to gather valueable hints concerning the completion of [5].

2. 7.-13. October, 2007: Oberwolfach: Workshop on 'Percolation'. Here, I gave a talk about [6]; useful discussion with Prof. J. Steiff helped to complete this paper.

3. 7.10.2008-13.13.2008: Greifswald: 'Geometry and Stochastics', During this conference, I gave a presentation about hyperbolic graphs and continued my work with Prof. Vadim Kaimanovich.

4. 12.October, 2009-17.10.2009: University of Lund, Seminar-Talk on [2, 3], see

   http://www.maths.lth.se/matstat/seminar/index09.html; During this week, Prof. Tatyana Turova and I worked on finding the right asymptotics of the integrated density of states (equivalently: the expected return probability) on inhomogeneous random graphs. We focused on the rank-1 case of the Bolobás-Riordan-Janson model.

6.2 Invitations of Guests to Graz

1. July 2007: Prof. Daniel Lenz, and Prof. I. Veselic visited Graz for one week. We worked on [8]. The collaboration with Prof. Lenz has led to my current employment at Jena university.

2. 4./5. July, 2009: St-Kathrein, Alp-Workshop. The invitation of all the speakers to the Alp-workshop was financed by the budget of the Project P18703 available for 'special tasks'. Also the left-over available budget concerning personnel cost, available due to my early finishing the project in April 2009 was used for this event.