Risk and Management: Goals and Perspective

Etymology: Risicare

Risk (Oxford English Dictionary): (Exposure to) the possibility of loss, injury, or other adverse or unwelcome circumstance; a chance or situation involving such a possibility.
Finance: The possibility that an actual return on an investment will be lower than the expected return.

Risk management: is the identification, assessment, and prioritization of risks followed by coordinated and economical application of resources to minimize, monitor, and control the probability and/or impact of unfortunate events or to maximize the realization of opportunities.
Risk management's objective is to assure uncertainty does not deflect the endeavor from the business goals.

Risk and Management: Goals and Perspective

Subject of risk managment:

- Identification of risk sources (determination of exposure)
- Assessment of risk dependencies
- Measurement of risk
- Handling with risk
- Control and supervision of risk
- Monitoring and early detection of risk
- Development of a well structured risk management system

Risk and Management: Goals and Perspective

Main questions addressed by strategic risk managment:

- Which are the strategic risks?
- Which risks should be carried by the company?
- Which instruments should be used to control risk?
- What resources are needed to cover for risk?
- What are the risk adjusted measures of success used as steering mechanisms?

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Examples: standard deviation, quantile of the loss distribution, ...

Types of risk

For an organization risk arises through events or activities which could prevent the organization from fulfilling its goals and executing its strategies.
Financial risk:

- Market risk
- Credit risk
- Operational risk
- Liquidity risk, legal (judicial) risk, reputational risk

The goal is to estimate these risks as precisely as possible, ideally based on the loss distribution (LD).

Regulation and supervision

1974: Establishment of Basel Committee on Banking Supervision (BCBS).
Risk capital depending on GD/LD.
Suggestions and guidelines on the requirements and methods used to *compute the risk capital*. Aims at *internationally accepted standards* for the computation of the risk capital and *statutory dispositions* based on those standards.
Control by the supervision agency.

1988 Basel I: International minimum capital requirements especially with respect to (w.r.t.) credit risk.

1996 Standardised models are formulated for the assessment of market risk with an option to use value at risk (VaR) models in larger banks

2007 Basel II: minimum capital requirements w.r.t. credit risk, market risk and operational risk, procedure of control by supervision agencies, market discipline[1].

2010 BASEL III - Improvement and further development of BASEL II w.r.t. applicability, operational risk und liquidity risk

[1] see http://www.bis.org

Assessment of the loss function

Loss operators
$V(t)$ - Value of portfolio at time t
Time unit Δt
Loss in time interval $[t, t + \Delta t]$: $L_{[t,t+\Delta t]} := -(V(t + \Delta t) - V(t))$
Discretisation of time: $t_n := n\Delta t$, $n = 0, 1, 2, \ldots$

$$L_{n+1} := L_{[t_n, t_{n+1}]} = -(V_{n+1} - V_n), \text{ where } V_n := V(n\Delta t)$$

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Let $Z_{n,i} := \ln S_{n,i}$, $X_{n+1,i} := \ln S_{n+1,i} - \ln S_{n,i}$
Let $w_{n,i} := \alpha_i S_{n,i}/V_n$, $i = 1, 2, \ldots, d$, be the relative portfolio weights.

Loss operator of an asset portfolio (cont.)

The following holds:

$$L_{n+1} := -\sum_{i=1}^{d} \alpha_i S_{n,i} \left(\exp\{X_{n+1,i}\} - 1 \right) =$$

$$-V_n \sum_{i=1}^{d} w_{n,i} \left(\exp\{X_{n+1,i}\} - 1 \right) =: l_n(X_{n+1})$$

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Linearisation $e^x = 1 + x + o(x^2) \sim 1 + x$ implies

$$L_{n+1}^{\Delta} = -V_n \sum_{i=1}^{d} w_{n,i} X_{n+1,i} =: l_n^{\Delta}(X_{n+1}),$$

where L_{n+1} ($L_{n+1]}^{\Delta}$) is the (linearised) loss function and l_n (l_n^{Δ}) is the (linearised) loss operator.

The general case

Let $V_n = f(t_n, Z_n)$ and $Z_n = (Z_{n,1}, \ldots, Z_{n,d})$, where Z_n is a vector of risk factors

Risk factor changes: $X_{n+1} := Z_{n+1} - Z_n$

$$L_{n+1} = -\left(f(t_{n+1}, Z_n + X_{n+1}) - f(t_n, Z_n) \right) =: l_n(X_{n+1}), \text{ where}$$

$$l_n(x) := -\left(f(t_{n+1}, Z_n + x) - f(t_n, Z_n) \right) \text{ is the loss operator.}$$

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The linearised loss:

$L^{\Delta}_{n+1} = -\left(f_t(t_n, Z_n)\Delta t + \sum_{i=1}^{d} f_{z_i}(t_n, Z_n) X_{n+1,i} \right)$,

where f_t and f_{z_i} are the partial derivatives of f.

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Value of ECO at time t: $C(t) = \max\{S(t) - K, 0\}$,
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Definition: A currency forward or an FX forward (FXF) is a contract between two parties to buy/sell an amount $\bar{V}$ of foreign currency at a future time T for a specified exchange rate $\bar{e}$. The party who is going to buy the foreign currency is said to hold a long position and the party who will sell holds a short position.

Example A bond portfolio

Let $B(t, T)$ be the price of the ZCB with maturity T at time $t < T$.
The *continuously compounded yield*, $y(t, T) := -\frac{1}{T-t} \ln B(t, T)$, represents the continuous interest rate which would have been dealt with at time t as being constant for the whole interval $[t, T]$.

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Portfolio value at time t_n:
$V_n = \sum_{i=1}^{d} \alpha_i B(t_n, T_i) = \sum_{i=1}^{d} \alpha_i exp\{-(T_i - t_n) Z_{n,i}\} = f(t_n, Z_n)$
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A bond portfolio (contd.)

$$l_{[n]}(x) = -\sum_{i=1}^{d} \alpha_i B(t_n, T_i) \left(exp\{Z_{n,i} \Delta t - (T_i - t_{n+1}) x_i\} - 1 \right)$$

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A long position over $(\bar{V})$ units of a FX forward with maturity T

$$\Longleftrightarrow$$

a long position over $\bar{V}$ units of a foreign zero-coupon bond (ZCB) with maturity T and a short position over $\bar{e}\bar{V}$ units of a domestic zero-coupon bond with maturity T.

A currency forward portfolio (contd.)

Assumptions:

Euro investor holds a long position of a USD/EUR forward over $\bar{V}$ USD.
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The linearized loss: $L^{\Delta}_{n+1} = -V_n(Z_{n,2}\Delta t + X_{n+1,1} - (T - t_{n+1})X_{n+1,2})$
where $X_{n+1,1} := \ln e(t_{n+1}) - \ln e(t_n)$ und
$X_{n+1,2} := y^f(t_{n+1}, T) - y^f(t_n, T)$

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Price of ECO at time $t < T$: $C = C(t, S, r, \sigma)$ (Black-Scholes model), where S is the price of the asset, r is the interest rate and σ is the volatility, all of them at time t.

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The greeks: C_t - theta, C_S - delta, C_r - rho, C_σ - Vega

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  Disadvantages: no difference between long and short positions, diversification effects are not condidered

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  Portfolio risk:

  $\Psi[\chi, w] = \max\{w_1 l_{[n]}(X_1), w_2 l_{[n]}(X_2), \ldots, w_N l_{[n]}(X_N)\}$

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An appropriate model (zB. Black-Scholes) is used to generate the option prices in the different scenarios.

- **Risk measures based on the loss distribution**

  Let $F_L := F_{L_{n+1}}$ be the loss distribution of L_{n+1}.

  The parameters of F_L will be estimated in terms of historical data, either directly or in terms of risk factors.

  1. **The standard deviation** $std(L) := \sqrt{\sigma^2(F_L)}$
     It is used frequently in portfolio theory.

     Disadvantages:

     - STD exists only for distributions with $E(F_L^2) < \infty$, not applicable to leptocurtic ("fat tailed") loss distributions;
     - gains and losses equally influence the STD.

     Example
     $L_1 \sim N(0,2)$, $L_2 \sim t_4$ (Student's t-distribution with $m = 4$ degrees of freedom)
     $\sigma^2(L_1) = 2$ and $\sigma^2(L_2) = \frac{m}{m-2} = 2$ hold
     However the probability of losses is much larger for L_2 than for L_1.
     Plot the logarithm of the quotient $\ln[P(L_2 > x)/P(L_1 > x)]$!

2. **Value at Risk** ($VaR_\alpha(L)$)

   Definition: Let L be the loss distribution with distribution function F_L. Let $\alpha \in (0, 1)$ be a given confindence level.

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Definition: Let $F \colon A \to B$ be an increasing function. The function $F^{\leftarrow} \colon B \to A \cup \{-\infty, +\infty\}, y \mapsto \inf\{x \in \mathbb{R} \colon F(x) \ge y\}$ is called *generalized inverse function* of F.

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If F is strictly monotone increasing, then $F^{-1} = F^{\leftarrow}$ holds.

Exercise: Compute $F^{\leftarrow}$ for $F \colon [0, +\infty) \to [0,1]$ with

$$F(x) = \begin{cases} 1/2 & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$

Value at Risk (contd.)

Definition: Let $F\colon \mathbb{R} \to \mathbb{R}$ be a (monotone increasing) distribution function and $q_\alpha(F) := \inf\{x \in \mathbb{R}\colon F(x) \geq \alpha\}$ be α-quantile of F.

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Example: Let $L \sim N(\mu, \sigma^2)$.
Then $VaR_\alpha(L) = \mu + \sigma q_\alpha(\Phi) = \mu + \sigma \Phi^{-1}(\alpha)$ holds, where Φ is the distribution function of a random variable $X \sim N(0, 1)$.

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Exercise: Consider a portfolio consisting of 5 pieces of an asset A. The today's price of A is $S_0 = 100$. The daily logarithmic returns are i.i.d.: $X_1 = \ln \frac{S_1}{S_0}$, $X_2 = \ln \frac{S_2}{S_1}, \ldots \sim N(0, 0.01)$. Let L_1 be the 1-day portfolio loss in the time interval (today, tomorrow).

(a) Compute $VaR_{0.99}(L_1)$.

(b) Compute $VaR_{0.99}(L_{100})$ and $VaR_{0.99}(L_{100}^{\Delta})$, where L_{100} is the 100-day portfolio loss over a horizon of 100 days starting with today. L_{100}^{Δ} is the linearization of the above mentioned 100-day PF-portfolio loss.

Hint: For $Z \sim N(0,1)$ use the equality $F_Z^{-1}(0.99) \approx 2.3$.

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If F_L **is discrete** the *generalized CVaR* is defined as follows:

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Then $CVaR_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_p(L)dp$ holds.