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Theorem: Let $(X_1, X_2)^T \sim t_2(0, \nu, R)$ be a random vector with a t-distribution and ν degrees of freedom, expectation 0 and linear correlation matrix R. For $R_{12} > -1$ we have

$$\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-R_{12}}}{\sqrt{1+R_{12}}}\right)$$

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The proof is similar to the proof of the analogous theorem about the Gaussian copulas.
Hint:

$$X_2|X_1 = x \sim \left(\frac{\nu+1}{\nu+x^2}\right)^{1/2} \frac{X_2 - \rho x}{\sqrt{1-\rho^2}} \sim t_{\nu+1}$$

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Corollary: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and a t-copula $C^t_{\nu,R}$ with ν degrees of freedom and and correlation matrix R. Then we have

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See McNeil et al. (2005) for a proof of the three last results.

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Clayton Copulas: $\phi(t) = (t^{-\theta} - 1)/\theta$, $\theta > 0$. Then

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Example Kendalls Tau for the Gumbel copula and the Clayton copula

Archimedian copulas (contd.)

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Let $\phi(t) = 1 - t$, $t \in [0,1]$. Then $\phi^{[-1]}(t) = \max\{1-t, 0\}$ and $C_\phi(u_1, u_2) := \phi^{[-1]}(\phi(u_1) + \phi(u_2)) = \max\{u_1 + u_2 - 1, 0\} = W(u_1, u_2)$. Thus the Fréchet lower bound is an Archimedian copula.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and an Archimedian copula C generated by ϕ. Then $\rho_\tau(X_1, X_2) = 1 + 4\int_0^1 \frac{\phi(t)}{\phi'(t)} dt$ holds.

See Nelsen 1999 for a proof.

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Multivariate Archimedean copulas

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Theorem: (Kimberling 1974)
Let $\phi\colon [0,1] \to [0,\infty]$ be a continuous, strictly monotone decreasing function with $\phi(0) = \infty$ and $\phi(1) = 0$. The function $C\colon [0,1]^d \to [0,1]$, $C(u) := \phi^{-1}(\phi(u_1) + \phi(u_2) + \ldots + \phi(u_d))$ is a copula for $d \geq 2$ iff ϕ^{-1} is completely monotone on $[0,\infty)$.

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Lemma: A function $\psi\colon [0,\infty) \to [0,\infty)$ is completely monotone with $\psi(0) = 1$ iff ψ is the Laplace-Stieltjes transform of some distribution function G on $[0,\infty)$, i.e. $\psi(s) = \int_0^\infty e^{-sx} dG(x)$, $s \geq 0$.

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- the generator function needs to fulfill quite restrictive technical assumptions