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- Let F_1, F_2 be the continuous marginal distributions of $(X_1, X_2)^T$ and let T_1, T_2 be strictly monotone functions on $[-\infty, \infty]$. Then the following equalities hold $\rho_\tau(X_1, X_2) = \rho_\tau(T_1(X_1), T_2(X_2))$ and $\rho_S(X_1, X_2) = \rho_S(T_1(X_1), T_2(X_2))$.

(See Embrechts et al., 2002).

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Definition: Let $(X_1, X_2)^T$ be a random vector with marginal distributions F_1 und F_2.

The coefficent $\lambda_U(X_1, X_2)$ *of the upper tail dependency* of $(X_1, X_2)^T$ is defined as $\lambda_U(X_1, X_2) = \lim_{u \to 1^-} P(X_2 > F_2^{\leftarrow}(u) | X_1 > F_1^{\leftarrow}(u))$, provided that the limit exists.

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If the limit exists and $\lambda_U > 0$ ($\lambda_L > 0$) we say that $(X_1, X_2)^T$ have an upper (a lower) tail dependence.

Tail dependency and survival copulas

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Definition: Let the copula C be the c.d.f. of a random vector $(U_1, U_2, \dots, U_d)$ with $U_i \sim U[0,1]$, $i = 1, 2, \dots, d$. The c.d.f. of $(1 - U_1, 1 - U_2, \dots, 1 - U_d)$ is called *survival copula* of C and is denoted by $\hat{C}$.

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Lemma: Let X be a random vector with multivariate tail distribution function $\bar{F}$ ($\bar{F}(x_1, x_2, \dots, x_d) := Prob(X_1 > x_1, X_2 > x_2, \dots, X_d > x_d)$) and marginal distributions F_i, $i = 1, 2, \dots, d$. Let $\bar{F}_i := 1 - F_i$, $i = 1, 2, \dots, d$. Then the following holds

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$$C_\theta^{\mathrm{Gu}}(u_1, u_2) = \exp\left(-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta\right]^{1/\theta}\right), \ \theta \geq 1$$

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The Clayton family of copulas:

$$C_\theta^{\mathrm{Cl}}(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{1/\theta},\ \theta > 0$$

We have $\lambda_U = 0$, $\lambda_L = 2^{-1/\theta}$.

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Definition: Let X be a d-dimensional random vector. Let $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d \times d}$ be constants, and let $\psi \colon [0, \infty) \to \mathbb{R}$ be a function such that $\phi_{X-\mu} = \psi(t^T \Sigma t)$ holds for the characteristic function $\phi_{X-\mu}$ of $X - \mu$. Then X is an elliptically distributed random vector with parameters μ, Σ, ψ. Notation: $X \sim E_d(\mu, \Sigma, \psi)$.

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Theorem:(Stochastic representation)
A d-dimensional random vector X is elliptically distributed, $X \sim E_d(\mu, \Sigma, \psi)$ with $rang(\Sigma) = k$, iff there exist a matrix $A \in \mathbb{R}^{d \times k}$, $A^T A = \Sigma$, a nonnegative r.v. R and a k-dimensional random vector U uniformly distributed on the unit ball $\mathcal{S}^{k-1} = \{z \in \mathbb{R}^k \colon z^T z = 1\}$, such that R and U are independent and $X \stackrel{d}{=} \mu + RAU$.

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Remark: An elliptically distributed random vector X ist *radial symmetric*, i.e. $X - \mu \stackrel{d}{=} \mu - X$.

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Let C_R^{Ga} be the copula of a d-dimensional normal distribution with correlation matrix R. Then $C_R^{Ga}(u) = \phi_R^d(\phi^{-1}(u_1), \ldots, \phi^{-1}(u_d))$ holds, where ϕ_R^d is the c.d.f. of a d-dimensional normal distribution with expected vector 0 and correlation matrix R, and ϕ^{-1} is the inverse of the standard normal distribution function.

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In the bivariate case we have:

$$C_R^{Ga}(u_1, u_2) = \int_{-\infty}^{\phi^{-1}(u_1)} \int_{-\infty}^{\phi^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{ \frac{-(x_1^2 - 2\rho x_1 x_2 + x_2^2)}{2(1-\rho^2)} \right\} dx_1 dx_2,$$

where $\rho \in (-1, 1)$.

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In the bivariate case ($d = 2$):

$$C^t_{\alpha,R}(u_1, u_2) = \int_{-\infty}^{t_\alpha^{-1}(u_1)} \int_{-\infty}^{t_\alpha^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \left\{1 + \frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{\alpha(1-\rho^2)}\right\}^{-\frac{\alpha+2}{2}} dx_1 dx_2$$

for $\rho \in (-1, 1)$. R_{12} is the linear correlation coefficient of the corresponding bivariate t_α-distribution for $\alpha > 2$.

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where $(U_1, U_2, \ldots, U_d)$ is a random vector with distribution function C.
For a radial symmetric copula $C = \hat{C}$ holds.

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where $(U_1, U_2, \ldots, U_d)$ is a random vector with distribution function C. For a radial symmetric copula $C = \hat{C}$ holds.

Example: Elliptical copulas are radial symmetric.

Further properties of copulas

Definition: (Radial symmetry)
A d-dimensional random vector X (or a d-variate distribution function) is called *radial symmetric around a*, for some $a \in \mathbb{R}^d$, iff $X - a \stackrel{d}{=} a - X$.

Example: An elliptically distributed random vector $X \sim E_d(\mu, \Sigma, \psi) \in \mathbb{R}^d$ is radial symmetric around μ.

Definition: (Radial symmetry of copulas)
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The Gumbel and Clayton Copulas are not radial symmetric. Why?

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$$c(u_1, u_2, \ldots, u_d) = \frac{\partial C(u_1, u_2, \ldots, u_d)}{\partial u_1 \partial u_2 \ldots \partial u_d}.$$

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Let C be the copula of a distribution F with marginal distributions F_1,...,F_d. By differentiating

$$C(u_1, \ldots, u_d) = F(F_1^{\leftarrow}(u_1), \ldots, F_d^{\leftarrow}(u_d))$$

we obtain the density c of C:

$$c(u_1, \ldots, u_d) = \frac{f(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))}{f_1(F_1^{-1}(u_1)) \ldots f_d(F_d^{-1}(u_d))}$$

where f is the density function of F, f_i are the marginal density functions, and F_i^{-1} are the inverse functions of F_i, for $1 \leq i \leq d$,

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Examples of exchangeable copulas:

Gumbel, Clayton, and also the Gaussian copula C_P^{Ga} and the t-Copula $C_{\nu,P}^t$, if P is an *equicorrelation matrix*, i.e. $R = \rho J_d + (1 - \rho) I_d$.
$J_d \in \mathbb{R}^{d \times d}$ is a matrix consisting only of ones, and $I_d \in \mathbb{R}^{d \times d}$ is the d-dimensional identity matrix.

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For bivariate exchangeable copulas we have:

$$P(U_2 \le u_2 | U_1 = u_1) = P(U_1 \le u_2 | U_2 = u_1).$$