Risk theory and risk management in actuarial science
Winter term 2017/18

6th work sheet

33. **Archimedian Copulas**

 (a) Show that for every $\theta \in \mathbb{R} \setminus \{0\}$ the function $\phi_\theta^{Fr}(t) = -\ln\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$ generates an Archmedian copula, the so-called Frank copula $C_\theta^{Fr}\colon [0,1]^2 \to [0,1]$. Check that the following equality holds $\forall u_1, u_2 \in [0,1]$:

$$C_\theta^{Fr}(u_1,u_2) = -\frac{1}{\theta}\ln\left(1+\frac{(\exp(-\theta u_1)-1)(\exp(-\theta u_2)-1)}{\exp(-\theta)-1}\right), \theta \in \mathbb{R}\setminus\{0\}\,.$$

 (b) Show that for every $\theta > 0$ and for every $\delta \geq 1$ the function $\phi_{\theta,\delta}^{GC}(t) = \theta^{-\delta}(t^{-\theta}-1)^\delta$ generates an Archmedian copula, the so-called *generalized Clayton copula* $C_{\theta,\delta}^{GC}\colon [0,1]^2 \to [0,1]$. Check that the following equality holds $\forall u_1, u_2 \in [0,1]$:

$$C_{\theta,\delta}^{GC}(u_1,u_2) = \{[(u_1^{-\theta}-1)^\delta + (u_2^{-\theta}-1)^\delta]^{1/\delta}+1\}^{-1/\theta},\ \theta \geq 0\,, \delta \geq 1\,.$$

 (c) Compute Kendall's tau ρ_τ as well as the coefficients λ_U, λ_L of the upper and lower tail dependency for the copulas C_θ^{Fr} and $C_{\theta,\delta}^{GC}$, respectively.

34. (a) Let $(X_1,X_2)^T$ be a t-distributed random vektor with ν degrees of freedom, expected value $(0,0)$ and linear correlation coefficient matrix $\rho \in (-1,1]$, i.e. $(X_1,X_2)^T \sim t_2(\vec{0},\nu,R)$ where R is 2×2 matrix with 1 on the diagonal and ρ outside the diagonal. Show that the following equality holds for $\rho > -1$:

$$\lambda_U(X_1,X_2) = \lambda_L(X_1,X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)$$

 Hint: Use the fact (no need to prove it!) that conditional on $X_1 = x$ the following holds

$$\left(\frac{\nu+1}{\nu+x^2}\right)^{1/2}\frac{X_2-\rho x}{\sqrt{1-\rho^2}} \sim t_{\nu+1}\,.$$

 Recall the stochatic representation of the bivariate t-distribution as $\mu + \sqrt{W}AZ$, where Z is bivariate standard normally distributed and W is such that $\frac{\nu}{W} \sim \chi_\nu^2$ while being independent on Z (cf. lecture).

 (b) Apply (a) to conclude that for a random vector with continuous marginal distributions $(X_1,X_2)^T$ and a t-copula $C_{\nu,R}^t$ with ν degrees of freedom and a correlation matrix R as in (a) the following equalities holds:

$$\lambda_U(X_1,X_2) = \lambda_L(X_1,X_2) = 2t_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)\,.$$

35. **Asymmetric bivariate copulas**

Let C_θ be any exchangeable bivariate copula. Then a parametric family of asymmetric copulas $C_{\theta,\alpha,\beta}$ is obtained by setting

$$C_{\theta,\alpha,\beta}(u_1,u_2) = u_1^{1-\alpha}u_2^{1-\beta}C_\theta(u_1^\alpha,u_2^\beta)\,,\ 0 \leq u_1,u_2 \leq 1, \qquad (1)$$

where $0 \leq \alpha,\beta \leq 1$. When C_θ is an Archimedian copula we refer to the copulas constructed by (1) as asymmetric bivariate Archimedian copulas.

Check that $C_{\theta,\alpha,\beta}$ defined as above is a copula by constructing a random vector with distribution function $C_{\theta,\alpha,\beta}$ and observing that its margins are standard uniform on $[0,1]$. Show that such a random vector (U_1,U_2) can be generated as follows:

(a) Generate a random pair (V_1, V_2) with distribution function C_θ.

(b) Generate, independently of V_1, V_2, two independent standard uniform variables $\bar{U}_1$ and $\bar{U}_2$.

(c) Set $U_1 = \max\{V_1^{1/\alpha}, \bar{U}_1^{1/(1-\alpha)}$ and $U_2 = \max\{V_2^{1/\beta}, \bar{U}_2^{1/(1-\beta)}$

Show that $C_{\theta,\alpha,\beta}$ is not exchangeable in general. What conditions should fulfill its parameters so as to obtain an exchangeable $C_{\theta,\alpha,\beta}$? Which copula results from $C_{\theta,\alpha,\beta}$ if $\alpha = \beta = 0$? Which copula results from $C_{\theta,\alpha,\beta}$ if $\alpha = \beta = 1$?

36. **Three-dimensional non-exchangeable Archimedian copulas**
Suppose that ϕ_1 and ϕ_2 are two *strict* generators of Archimedian copulas, i.e. ϕ_1 and ϕ_2 are completely monotonic decreasing functions for which the pseudo-inverse function coincides with the ordinary inverse function. Consider

$$C(u_1, u_2, u_3) = \phi_2^{-1}(\phi_2 \circ \phi_1^{-1}(\phi_1(u_1) + \phi_1(u_2)) + \phi_2(u_3)). \tag{2}$$

If ϕ_1^{-1}, ϕ_2^{-1} and $\phi_2 \circ \phi_1^{-1}$ are completely monotonic decreasing functions mapping $[0, \infty]$ to $[0, \infty]$, then C defined by (2) is a copula.

Let (U_1, U_2, U_3) be a random vector with distribution function C as defined by (2).

(a) Show that if $\phi_1 \neq \phi_2$, then only U_1 and U_2 are exchangeable, i.e. $(U_1, U_2, U_3) \stackrel{d}{=} (U_2, U_1, U_3)$, but no other swapping of subscritps is possible. Show moreoever that if $\phi_1 = \phi_2$, than C is an exchangeable copula.

(b) Show that all bivariate margins of C defined by (2) are themselves Archimedian copulas; the margins C_{13} (obtained by components 1 and 3) and C_{23} (obtained by components 2 and 3) have generator ϕ_2, whereas the margin C_{12} (obtained by components 1 and 2) has generator ϕ_1.

37. **Equivalent threshold models**
Let $X = (X_1, X_2, \ldots, X_m)'$ be an m-dimensional random vector and let $D \in \mathrm{I\!R}^{\mathrm{m}\times\mathrm{n}}$ be a deterministic matrix with elements d_{ij} such that for every i, $1 \le i \le m$, the elements of the i-th row form a set of increasing thresholds satisfying $d_{i,1} < d_{i2} \ldots < d_{in}$. Introduce additionally $d_{i0} = -\infty$, $d_{i,n+1} = +\infty$ and set

$$S_i = j \iff d_{ij} < X_i \le d_{i,j+1}\,, \text{for } j \in \{0, \ldots, n\}, i \in \{1, \ldots, m\}.$$

Then (X, D) is said to define a *threshold model for the state vector* $S = (S_1, \ldots, S_m)'$. We refere to X as the *vector of critical variables* and denote its marginal distribution functions by $F_i(x) = P(X_i \le x)$, for $i \in \{1, 2, \ldots, m\}$. The i-th row of D contains the critical thresholds for firm i. By definition, default (corresponding to event $S_i = 0$) occurs iff $X_i \le d_{i1}$, thus the default probability of company i is given by $\bar{p}_i := F_i(d_{i1})$. Let Y_i be the default indicator of company i, i.e. $Y_i \in \{0, 1\}$ with $Y_i = 1$ iff company 1 defaults, hence $Prob(Y_i = 1) = \bar{p}_i$ and $Prob(Y_i = 0) = 1 - \bar{p}_i$, for $1 \le i \le m$. We denote by $\rho(Y_i, Y_j)$ the *default correlation* of two firms $i \neq j$; this quantity depends on $E(Y_i, Y_j)$ (how?) which in turn depends on the joint distribution of (X_i, X_j), and hence on the copula of $(X_1, X_2, \ldots, X_m)'$. (Notice that in general the latter is not fully determined by the *asset correlation* $\rho(X_i, X_j)$.)

Two threshold models (X, D) and $(\tilde{X}, \tilde{D})$ for the state vectors S and $\tilde{S}$, respectively, are called equivalent, iff S and $\tilde{S}$ have the same probability distribution.

Show that two threshod models (X, D) and $(\tilde{X}, \tilde{D})$ with state vectors S and $\tilde{S}$, respectively, are equivalent if the following conditions hold:

(a) The marginal distributions of the random vectors S and $\tilde{S}$ coincide, i.e. $P(S_i = j) = P(\tilde{S}_i = j)$, for all $j \in \{1, \ldots, n\}$, $i \in \{1, \ldots, m\}$.

(b) X and $\tilde{X}$ admit the same copula C.