Risk theory and risk management in actuarial science
Winter term 2017/18

5th work sheet

28. Construct two random vectors $(X_1, X_2)^T$ and $(Y_1, Y_2)^T$ with different joint distributions $F_{(X_1,X_2)}$, $F_{(Y_1,Y_2)}$, respectively, such that

    (a) the variables X_1, X_2, Y_1, Y_2 are standard normally distributed, i.e. $X_1, X_2, Y_1, Y_2 \sim N(0,1)$,

    (b) the two X-variables and the two Y-variables are uncorrelated, respectively, i.e. $\rho_L(X_1, X_2) = 0$, $\rho_L(Y_1, Y_2) = 0$, and

    (c) the α-quantiles of the corresponding sums are different, i.e. $F^{\leftarrow}_{X_1+X_2}(\alpha) \neq F^{\leftarrow}_{Y_1+Y_2}(\alpha)$ holds for some $\alpha \in (0,1)$, where $F_{X_1+X_2}$, $F_{Y_1+Y_2}$ are the distributions of X_1+X_2 and Y_1+Y_2, respectively.

    Conclude that in general it is not possible to draw conclusions about the loss of a portfolio if the loss distributions of the single assets in portfolio and their mutual linear correlation coefficients are known.

    Hint: Choose (X_1, X_2) to be bivariate standard normally distributed, i.e. $(X_1, X_2) \sim N_2(0, I_2)$, where 0 denotes the zero vector in $\mathbb{R}^2$ and I_2 denotes the identity matrix in $\mathbb{R}^{2\times 2}$. Choose Y_1 to be standard normally distributed, $Y_1 \sim N(0,1)$, and set $Y_2 := VY_1$, where V is a discrete random variable independent on Y_1 with values 1 and -1 taken with probability 1/2 each.

29. (Co-monotonicity and anti-monotonicity)

    (a) Let Z be a random variable with continuous cumulative distribution function F, $Z \sim F$. Let f_1, f_2 be to monotone increasing functions on $\mathbb{R}$ and let f_3 be a monotone decreasing function on $\mathbb{R}$. Let $X_i = f_i(Z)$, for $i = 1, 2, 3$. Show that the Fréchet upper bound M is the copula of (X_1, X_2) and the Fréchet lower bound W is the copula of (X_1, X_3).

    (b) Let W be the copula of the random vector (X_1, X_2) with continuous marginal distributions F_1 and F_2, respectively. Show that $X_2 \overset{a.s.}{=} T(X_1)$ with $T = F_2^{\leftarrow} \circ (1 - F_1)$.

    (c) Let M be the copula of the random vector (X_1, X_2) with continuous marginal distributions F_1 and F_2, respectively. Show that $X_2 \overset{a.s.}{=} T(X_1)$ with $T = F_2^{\leftarrow} \circ F_1$.

30. Prove the following equalities for the rank correlations of a random vector $(X_1, X_2)^T$ with continuous marginal distributions and unique copula C:

    $\rho_\tau(X_1, X_2) = 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1$,

    $\rho_S(X_1, X_2) = 12 \int_0^1 \int_0^1 (C(u_1, u_2) - u_1 u_2) du_1 du_2 = 12 \int_0^1 \int_0^1 C(u_1, u_2) du_1 du_2 - 3$.

31. Consider the coefficients of the tail dependence of a random vector $(X_1, X_2)^T$ with continuous marginal distributions and unique copula C. Show that the following equalities hold, provided that the corresponding limits exits:

    $\lambda_U(X_1, X_2) = \lim_{u\to 1^-} \frac{1-2u+C(u,u)}{1-u}$ and $\lambda_L(X_1, X_2) = \lim_{u\to 0^+} \frac{C(u,u)}{u}$.

32. The Gumbel family C_θ^{Gu} and the Clayton family C_θ^{Cl} are two one-parametric families of copulas given as

$$C_\theta^{\mathrm{Gu}}(u_1, u_2) := \exp\left(- \left[(-\ln u_1)^\theta + (-\ln u_2)^\theta \right]^{1/\theta} \right), \ \theta \geq 1, \text{ and}$$

$$C_\theta^{\mathrm{Cl}}(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}, \ \theta > 0.$$

    (a) Compute Kendall's tau ρ_τ as well as the coefficients λ_U, λ_L of the upper and lower tail dependence for the copulas C_θ^{Gu}, C_θ^{Cl}, respectively.

(b) The independence copula Π is given by $\Pi(u_1, u_2) := u_1 u_2$, for $(u_1, u_2) \in [0,1]^2$. Show that C_θ^{Gu} tends to the independence copula Π if θ tends to 1 and to the upper Fréchet bound M if θ tends to infinity. In this case we say that *the lower limit of the Gumbel copula is the independence copula* Π *and its upper limit is the Fréchet upper bound* M. Analogously show that the lower limit of the Clayton copula is the independence copula Π for $\theta \to 0^+$ and its upper limit is the Fréchet upper bound M for $\theta \to +\infty$. Now considerer an extension of the Clayton copula C_θ^{Cl} for $\theta \in [-1, 0)$, defined as an Archimedian copula with generator $\phi_\theta(t) = \frac{1}{\theta}(t^{-\theta} - 1)$ for $t \in (0, 1]$ and $\phi_\theta(0) = +\infty$. Show that for $\theta = -1$ the Clayton copula C_{-1}^{Cl} coincides with the Fréchet lower bound W.