Risk theory and risk management in actuarial science
Winter term 2017/18

4th work sheet

21. Let the random variables X_i, $i = 1, 2$, be such that $X_1 \sim Exp(\lambda)$ and $X_2 = X_1$, where $Exp(\lambda)$ is the exponential distribution with parameter λ. Consider the strictly increasing functions $t_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, with $t_1(x) = x$ and $t_2(x) = x^2$. Show that the following equalities for the linear correlation coefficient ρ_L hold:
$$\rho_L(X_1, X_2) = 1 \text{ and } \rho_L\Big(t_1(X_1), t_2(X_2)\Big) = \frac{2}{\sqrt{5}}\,.$$

22. Let the random variables X_i, $i = 1, 2$, be such that $X_1 \sim Exp(\lambda)$ and $X_2 = X_1^2$, where $Exp(\lambda)$ is the exponential distribution with parameter λ. Determine the coefficients of the lower and the upper tail dependence $\lambda_L(X_1, X_2)$, $\lambda_U(X_1, X_2)$, respectively, and conclude that X_1 and X_2 have both a lower and an upper tail dependence. Compute also the coefficient of the linear correlation $\rho_L(X_1, X_2)$, compare the three computed dependence measures and comment on your results.

23. (Properties of the generalized inverse function)

    Let $h\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $h(\mathbb{R}) = \mathbb{R}$ and let $h^{\leftarrow}\colon \mathbb{R} \to \mathbb{R}$ be generalized inverse function of h. Show that the following statements hold.

    (a) $h^{\leftarrow}$ is a monotone increasing left-continuous function.
    (b) h is continuous if and only if $h^{\leftarrow}$ is strictly monotone increasing .
    (c) h is strictly monotone increasing if and only if $h^{\leftarrow}$ is continuous.
    (d) $h^{\leftarrow}(h(x)) \le x$
    (e) If h is strictly monotone increasing then $h^{\leftarrow}(h(x)) = x$.
    (f) h is continuous if and only if $h(h^{\leftarrow}(y)) = y$.

24. (A coherent premium principle)

    Consider two constants $p > 1$ and $\alpha \in [0, 1)$. Let $(\Omega, \mathcal{F}, P)$ be some fixed probability space and $\mathcal{M}$ be the set of all random variables L on $(\Omega, \mathcal{F})$ for which $E(|L|^p)^{1/p}$ is finite, i.e. $E(|L|^p)^{1/p} < \infty$. Define a risk measure $\rho_{\alpha,p} := E(L) + \alpha(||(L - E(L))^+||_p$ on $\mathcal{M}$, where $||X||_p := E(|X|^p)^{1/p}$ is the L^p-norm of the positive part of the centered random variable $X - E(X)$ for any random variable $X \in \mathcal{M}$. Show that $\rho_{\alpha,p}$ is a coherent risk measure for any $p > 1$ and any $\alpha \in [0, 1)$. So we get a whole family of coherent risk measures $\rho_{\alpha,p}$ for $p > 1$ and $\alpha \in [0, 1)$. How do the parameters α and p influence $\rho_{\alpha,p}$? Which parameter values lead to more "conservative" risk measures?

25. (Generalized scenarios as coherent risk measures)

    Denote by $\mathcal{P}$ a set of probability measures on some underlying measurable space $(\Omega, \mathcal{F})$ and set
$$\mathcal{M}_{\mathcal{P}} := \{L\colon L \text{ is a r.v. on } (\Omega, \mathcal{F}),\ E^Q(|L|) < \infty \text{ for all } Q \in \mathcal{P}\}\,,$$
    where $E^Q(X)$ denotes the expected value of a random variable X under the probability measure Q. Then *the risk measure induced by the set of generalized scenarios* $\mathcal{P}$ is the mapping $\rho_{\mathcal{P}}\colon \mathcal{M}_{\mathcal{P}} \to \mathbb{R}$ such that $\rho_{\mathcal{P}}(L) := \sup\{E^Q(L)\colon Q \in \mathcal{P}\}$. Show that $\rho_{\mathcal{P}}$ is coherent on $\mathcal{M}_{\mathcal{P}}$ for any set $\mathcal{P}$ of probability measures on $\mathcal{M}_{\mathcal{P}}$. Interprete the scenario based risk measures (cf. lecture) as a risk measure generalized by an appropriately defined set of probability measures on appropriately defined discrete probability spaces[1].

[1] It can be shown that in the case of discrete probability spaces any coherent risk measure is induced by some set of generalized scenarios as described above, see Proposition 6.11 in *A.J. McNeil, R. Frey and P. Embrechts, Quantitative Risk Management: Concepts, Techniques amd Tools , Princeton Unversity Press, 2005.*

26. Show that the Fréchet lower bound W_d is not a copula for $d \geq 3$.

Hint: Show that the rectangle inequality

$$\sum_{k_1=1}^{2} \sum_{k_2=1}^{2} \cdots \sum_{k_d=1}^{2} (-1)^{k_1+k_2+\ldots+k_d} W_d(u_{1k_1}, u_{2k_2}, \ldots, u_{dk_d}) \geq 0\,,$$

where $(a_1, a_2, \ldots, a_d)$, $(b_1, b_2, \ldots, b_d) \in [0,1]^d$ with $a_k \leq b_k$ and $u_{k1} = a_k$ und $u_{k2} = b_k$ for all $k \in \{1, 2, \ldots, d\}$, is violated if $d \geq 3$ and $a_i = \frac{1}{2}$, $b_i = 1$, for $i = 1, 2, \ldots, d$.

27. Let X_i, $i = 1, 2$, be two lognormally distributed random variables with $X_1 \sim Lognormal(0,1)$ und $X_2 \sim Lognormal(0, \sigma^2)$, $\sigma > 0$. Compute $\rho_{L,min}(X_1, X_2)$ and $\rho_{L,max}(X_1, X_2)$ in dependence of σ and compare their values for different values of $\sigma > 0$. What can you say about the copula of (X_1, X_2) in each of the cases? Plot the graphs of $\rho_{L,min}(X_1, X_2)$ and $\rho_{L,max}(X_1, X_2)$ as functions of σ and comment on the behaviour of these functions for $\sigma \to +\infty$?

Hint: Consider $X_1 = \exp(Z)$ and $X_2 = \exp(\sigma Z)$ or $X_2 = \exp(-\sigma Z)$ for a standard normally distributed random variable Z.