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Any (long-short) portfolio in $\mathcal{P}$ is uniquelly determined by its weight vector $w = (w_i) \in \mathbb{R}^d$ with $\sum_{i=1^d} |w_i| = 1$. $w_i > 0$ ($w_i < 0$) represents a long (short) investment.

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Let $\mathcal{P}_m$ be the family of portfolios in $\mathcal{P}$ with $E(Z(w)) = m$, for some $m \in \mathbb{R}$, $m > 0$.
$\mathcal{P}_m := \{w = (w_i) \in \mathbb{R}^d, \sum_{i=1}^d |w_i| = 1, w^T \mu = m\}$

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For a risk emasure ρ the *mean-ρ portfolio optimization model* is:

$$\min_{w \in \mathcal{P}_m} \rho(Z(w)) \tag{1}$$

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If $Cov(x) = \Sigma$ and the weights are nonnegative (long-only portfolio) we get the Markovitz portfolio optimization model (Markowitz 1952):

$$\begin{array}{ll} \min_{w} & w^T \Sigma w \\ \text{s.t.} & \\ & w^T \mu = m \\ & \sum_{i=1}^{d} |w_i| = 1 \end{array}$$

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If $\rho = VaR_\alpha$, $\alpha \in (0,1)$ we get the *mean-VaR pf. optimization model*

$$\min_{w \in \mathcal{P}_m} VaR_\alpha(Z(w)).$$

Question: What is the relationship between these three portfolio optimization models?

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Theorem: (Embrechts et al., 2002)
Let M be the set of returns of the portfolii in $\mathcal{P} := \{w = (w_i) \in \mathbb{R}^d, \sum_{i=1}^{d} |w_i| = 1\}$. Let the asset returns $X = (X_1, X_2, \ldots, X_d)$ be elliptically distributed, $X = (X_1, X_2, \ldots, X_d) \sim E_d(\mu, \Sigma, \psi)$ for some $\mu \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d\times d}$ and $\psi \colon \mathbb{R} \to \mathbb{R}$. Then VaR_α ist coherent in M, for any $\alpha \in (0.5, 1)$.

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Theorem: (Embrechts et al., 2002)
Let $X = (X_1, X_2, \ldots, X_d) = \mu + AY$ be elliptically distributed with $\mu \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times k}$ and a spherically distributed vector $Y \sim S_k(\psi)$. Assume that $0 < E(X_k^2) < \infty$ holds $\forall k$. If the risk measure ρ has the properties (C1) and (C3) and $\rho(Y_1) > 0$ for the first component Y_1 of Y, then

$$\arg\min\{\rho(Z(w))\colon w \in \mathcal{P}_m\} = \arg\min\{var(Z(w))\colon w \in \mathcal{P}_m\}$$

Copulas: Definition and basic properties

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Equivalently, a copula C is a function $C\colon [0,1]^d \to [0,1]$, with the following properties:

1. $C(u_1, u_2, \ldots, u_d)$ is mon. increasing in each variable u_i, $1 \le i \le d$.
2. $C(1, 1, \ldots, 1, u_k, 1, \ldots, 1) = u_k$ for any $k \in \{1, \ldots, d\}$ and $\forall u_k \in [0,1]$.
3. The *rectangle inequality* holds $\forall (a_1, a_2, \ldots, a_d) \in [0,1]^d$, $\forall (b_1, b_2, \ldots, b_d) \in [0,1]^d$ with $a_k \le b_k$, $\forall k \in \{1, 2, \ldots, d\}$:

$$\sum_{k_1=1}^{2} \cdots \sum_{k_d=1}^{2} (-1)^{k_1+k_2+\ldots+k_d} C(u_{1k_1}, u_{2k_2}, \ldots, u_{dk_d}) \ge 0\,,$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$.

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Remark: The k-dimensional marginal distributions of a d-dimensional copula are k-dimensional copulas, for all $2 \le k \le d$.

Lemma: Let $h \colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $h(\mathbb{R}) = \mathbb{R}$ and $h^{\leftarrow} \colon \mathbb{R} \to \mathbb{R}$ be the generalized inverse function of h. Then the following statements hold:

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Lemma: Let X be a r.v. with continuous distribution function F. Then $P\left(F^{\leftarrow}(F(x)) = x\right) = 1$, i.e. $F^{\leftarrow}(F(X)) \overset{a.s.}{=} X$