Properties of elliptical distributions

Theorem:

Let $X \sim E_k(\mu, \Sigma, \psi)$. X has the following properties:

- Linear combinations

  For $B \in \mathbb{R}^{k \times d}$ and $b \in \mathbb{R}^k$ we have:

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- Marginal distributions

  Set $X^T = \left({X^{(1)}}^T, {X^{(2)}}^T \right)$ for ${X^{(1)}}^T = (X_1, X_2, \dots, X_n)^T$ and ${X^{(2)}}^T = (X_{n+1}, X_{n+2}, \dots, X_k)^T$ and analogously set $\mu^T = \left({\mu^{(1)}}^T, {\mu^{(2)}}^T \right)$ as well as $\Sigma = \begin{pmatrix} \Sigma^{(1,1)} & \Sigma^{(1,2)} \\ \Sigma^{(2,1)} & \Sigma^{(2,2)} \end{pmatrix}$. Then $X_1 \sim E_n\left(\mu^{(1)}, \Sigma^{(1,1)}, \psi \right)$ and $X_2 \sim E_{k-n}\left(\mu^{(2)}, \Sigma^{(2,2)}, \psi \right)$.

Properties of elliptical distributions (contd.)

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- Conditional distributions

  Assume that Σ is nonsingular. Then

  $$X^{(2)} \Big| X^{(1)} = x^{(1)} \sim E_{d-k}\left(\mu^{(2,1)}, \Sigma^{(22,1)}, \tilde{\psi}\right) \text{ where}$$

  $$\mu^{(2,1)} = \mu^{(2)} + \Sigma^{(2,1)}\left(\Sigma^{(1,1)}\right)^{-1}\left(x^{(1)} - \mu^{(1)}\right) \text{ and}$$

  $$\Sigma^{(22,1)} = \Sigma^{(2,2)} - \Sigma^{(2,1)}\left(\Sigma^{(1,1)}\right)^{-1}\Sigma^{(1,2)}.$$

  Typically $\tilde{\psi}$ is a different characteristic generator than the original ψ (see Fang, Katz and Ng 1987).

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- Quadratic forms

  If Σ is nonsingular, then $D^2 = (X-\mu)^T\Sigma^{-1}(X-\mu) \sim R^2$, where R is the nonnegative r.v. in the stochastic representation $Y = RS$ of the spherical distribution Y with $S \sim U\left(\mathcal{S}^{(d-1)}\right)$ and $X = \mu + AY$.

  The random variable D is called *Mahalanobis distance*.

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- Convolutions

  Let $X \sim E_k(\mu, \Sigma, \psi)$ and $Y \sim E_k(\tilde{\mu}, \Sigma, \tilde{\psi})$ be two independent random vectors. Then $X + Y \sim E_k(\mu + \tilde{\mu}, \Sigma, \bar{\psi})$ where $\bar{\psi} = \psi\tilde{\psi}$.

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Important: $X \sim E_k(\mu, I_k, \psi)$ does not imply that the components of X are independent. The components of X are independent iff X is multivariate normally distributed with the unit matrix as a covariance matrix.

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Convex risk measures

Consider the property:

(C5) Convexity:
$\forall X_1, X_2 \in M$, $\forall \lambda \in [0,1]$
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Observation: VaR is not coherent in general.

Let the probability measure P be defined by some continuous or discrete probabilty distribution F.

$VaR_\alpha(F) = F^{\leftarrow}(\alpha)$ has the properties (C1), (C3) and (C4), whereas the subadditivity (C2) is not fulfilled in general.

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Example: Let the probability measure P be defined by the binomial distribution $B(p, n)$ for $n \in \mathbb{N}$, $p \in (0, 1)$. We show that $VaR_\alpha(B(p, n))$ is not subadditive.

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Theorem: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $M \subseteq L^{(0)}(\Omega, \mathcal{F}, P)$ be the set of the random variables with a continuous distribution in $(\Omega, \mathcal{F}, P)$. $CVaR_\alpha$ is a coherent risk measure in M, $\forall \alpha \in (0, 1)$.

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Sketch of the proof:

(C1),(C3), (C4) follow from $CVaR_\alpha(F) = \frac{1}{1-\alpha} \int_\alpha^1 Var_p(F) dp$.

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To show (C2) observe that for a sequence of i.i.d. r.v. L_1, L_2, ..., L_n with order statistics $L_{1,n} \geq L_{2,n} \geq \ldots \geq L_{n,n}$ and for any $m \in \{1,2,\ldots,n\}$

$$\sum_{i=1}^{m} L_{i,n} = \sup\{L_{i_1} + L_{i_2} + \ldots + L_{i_m} : 1 \leq i_1 < \ldots < i_m \leq n\} \text{ holds.}$$