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Theorem: (Characterisation of $MDA(H_\gamma)$)

- $F \in MDA(H_\gamma)$ with $\gamma > 0 \iff F \in MDA(\Phi_\alpha)$ with $\alpha = 1/\gamma > 0$.
- $F \in MDA(H_0) \iff F \in MDA(\Lambda)$.
- $F \in MDA(H_\gamma)$ with $\gamma < 0 \iff F \in MDA(\Psi_\alpha)$ with $\alpha = -1/\gamma > 0$.

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Theorem: ($MDA(\Phi_\alpha)$, Gnedenko 1943)
$F \in MDA(\Phi_\alpha)$ $(\alpha > 0) \iff \bar{F} \in RV_{-\alpha}$ $(\alpha > 0)$.
If $F \in MDA(\Phi_\alpha)$, then $\lim_{n\to\infty} a_n^{-1} M_n = \Phi_\alpha$ with $a_n = F^{\leftarrow}(1 - n^{-1})$.

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Examples: The following distributions belong to $MDA(\Phi_\alpha)$:

- Pareto: $F(x) = 1 - x^{-\alpha}$, $x > 1$, $\alpha > 0$.
- Cauchy: $f(x) = (\pi(1 + x^2))^{-1}$, $x \in \mathbb{R}$.
- Student: $f(x) = \frac{\Gamma((\alpha+1)/2)}{\sqrt{\alpha\pi}\,\Gamma(\alpha/2)(1+x^2/\alpha)^{(\alpha+1)/2}}$, $\alpha \in \mathbb{N}$, $x \in \mathbb{R}$.
- Loggamma: $f(x) = \frac{\alpha^\beta}{\Gamma(\beta)}(\ln x)^{\beta-1} x^{-\alpha-1}$, $x > 1$, $\alpha, \beta > 0$.

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Theorem: ($MDA(\Psi_\alpha)$, Gnedenko 1943)
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If $F \in MDA(\Psi_\alpha)$, then $\lim_{n\to\infty} a_n^{-1}(M_n - x_F) = \Psi_\alpha$ with $a_n = x_F - F^{\leftarrow}(1 - n^{-1})$.

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Example: Let $X \sim U(0,1)$. it holds $X \in MDA(\Psi_1)$ with $a_n = 1/n$, $n \in \mathbb{N}$.

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Theorem: ($MDA(\Lambda)$)
Let F be a distribution function with right endpoint $x_F \leq \infty$.
$F \in MDA(\Lambda)$ holds iff there exists a $z < x_F$ such that F can be represented as

$$\bar{F}(x) = c(x)exp\left\{ -\int_z^x \frac{g(t)}{a(t)}dt \right\}, \forall x, z < x \leq x_F,$$

where the function $c(x)$ and $g(x)$ fulfill $\lim_{x\uparrow x_F} c(x) = c > 0$ and $\lim_{t\uparrow x_F} g(t) = 1$, and $a(t)$ is a positive absolutely continuous function with $\lim_{t\uparrow x_F} a'(t) = 0$.

Characterisations of MDAs (contd.)

Theorem: ($MDA(\Lambda)$, alternative Characterisation)
A distribution function F belongs to $MDA(\Lambda)$ iff there exists a a positive function $\tilde{a}$ such that

$$\lim_{x \uparrow x_F} \frac{\bar{F}(x + u\tilde{a}(x))}{\bar{F}(x)} = e^{-u}, \forall u \in \mathbb{R}$$

A possible choice for $\tilde{a}$ is $\tilde{a}(x) = a(x)$ with $a(x) := \int_x^{x_F} \frac{\bar{F}(t)}{\bar{F}(x)} dt$.

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Definition: The function $a(x)$ above is called *mean excess function* and it can be alternatively represented as

$$a(x) := E(X - x | X > x), \forall x \leq x_F \, .$$

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Examples: The following distributions belong to $MDA(\Lambda)$:

- Normal: $F(x) = (2\pi)^{-1/2}\exp\{-x^2/2\}$, $x \in \mathbb{R}$.
- Exponential: $f(x) = \lambda^{-1}\exp\{-\lambda x\}$, $x > 0$, $\lambda > 0$.
- Lognormal: $f(x) = (2\pi x^2)^{-1/2}\exp\{-(\ln x)^2/2\}$, $x > 0$.
- Gamma: $f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\exp\{-\beta x\}$, $x > 0$, $\alpha, \beta > 0$.

Graphical methods for the investigation of the right tail of the distribution

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  Let $x_{n,n} \leq x_{n-1,n} \leq \ldots \leq x_{1,n}$ be a sorted sample of X_1, X_2,. . ., X_n. qq-plot: $\{(x_{k,n}, F^{\leftarrow}(\frac{n-k+1}{n+1})) \colon k = 1, 2, \ldots, n\}$.

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  If the assumption is plausible then the qq-plot is similar to the graph of a linear function. This property holds also if the reference distribution and the real distribution do not coincide but are of the same type.

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  Rule of thumb: the larger the quantile the heavier the tails of the distribution!

The Hill estimator

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Goal: Estimate α!

Theorem: (Theorem of Karamata)
Let L be a slowly varying locally bounded function on $[x_0, +\infty)$ for some $x_0 \in \mathbb{R}$. Then the following holds:

(a) For $\kappa > -1$: $\int_{x_o}^{x} t^{\kappa} L(t) dt \sim K(x_0) + \frac{1}{\kappa+1} x^{\kappa+1} L(x)$ for $x \to \infty$, where $K(x_0)$ is a constant depending on x_0.

(b) For $\kappa < -1$: $\int_{x}^{+\infty} t^{\kappa} L(t) dt \sim -\frac{1}{\kappa+1} x^{\kappa+1} L(x)$ for $x \to \infty$.

Proof in Bingham et al. 1987.

Application of Karamata's theorem

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The theorem of Karamata implies: $E\left(\ln(X) - \ln(u) \middle| \ln(X) > \ln(u)\right) =$

$$\lim_{u\to\infty} \frac{1}{\bar{F}(u)} \int_u^\infty (\ln x - \ln u) dF(x) = \alpha^{-1}. \tag{8}$$

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For the empirical distribution $F_n(x) = \frac{1}{n} \sum_{k=1}^n I_{[x_k, \infty)}(x)$ and a large threshold $x_{k,n}$ depending on the sample $x_{n,n} \leq x_{n-1,n} \leq \ldots \leq x_{1,n}$ we get:

$$E\left(\ln(X) - \ln(x_{k,n}) \middle| \ln(X) > \ln(x_{k,n})\right) \approx$$

$$\frac{1}{\bar{F}_n(x_{k,n})} \int_{X_{k,n}}^\infty (\ln x - \ln x_{k,n}) dF_n(x) = \frac{1}{k-1} \sum_{j=1}^{k-1} (\ln x_{j,n} - \ln x_{k,n}).$$

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If $k = k(n) \to \infty$ and $k/n \to 0$, then $x_{k,n} \to \infty$ for $n \to \infty$, and (8) implies:

$$\lim_{n \to \infty} \frac{1}{k-1} \sum_{j=1}^{k-1} (\ln x_{j,n} - \ln x_{k,n}) \stackrel{d}{=} \alpha^{-1}$$

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Thus the following Hill estimator is consistent:

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Given an estimator $\hat{\alpha}_{k,n}^{(H)}$ of α we get tail and quantile estimators as follows:

$$\hat{\bar{F}}(x) = \frac{k}{n} \left(\frac{x}{x_{k,n}} \right)^{-\hat{\alpha}_{k,n}^{(H)}} \text{ and } \hat{q}_p = \hat{F}^{\leftarrow}(p) = \left(\frac{n}{k}(1-p) \right)^{-1/\hat{\alpha}_{k,n}^{(H)}} x_{k,n}.$$

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Definition: (The generalized Pareto distribution (GPD))
The standard GPD denoted by G_γ:

$$G_\gamma(x) = \begin{cases} 1 - (1+\gamma x)^{-1/\gamma} & \text{für } \gamma \neq 0 \\ 1 - \exp\{-x\} & \text{für } \gamma = 0 \end{cases}$$

where $x \in D(\gamma)$

$$D(\gamma) = \begin{cases} 0 \leq x < \infty & \text{für } \gamma \geq 0 \\ 0 \leq x \leq -1/\gamma & \text{für } \gamma < \ 0 \end{cases}$$

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Notice that $G_0 = \lim_{\gamma \to 0} G_\gamma$.

Let $\nu \in \mathbb{R}$ and $\beta > 0$. The GPD with parameters γ, ν, β is given by the following distribution function

$$G_{\gamma,\nu,\beta} = 1-(1+\gamma\frac{x-\nu}{\beta})^{-1/\gamma}$$

where $x \in D(\gamma,\nu,\beta)$ and

$$D(\gamma,\nu,\beta) = \begin{cases} \nu \leq x < \infty & \text{für } \gamma \geq 0 \\ \nu \leq x \leq \nu-\beta/\gamma & \text{für } \gamma < 0 \end{cases}$$

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Theorem: Let $\gamma \in \mathbb{R}$. The following statements are equiavlent:

(i) $F \in MDA(H_\gamma)$

(ii) There exists a positive measurable function $a(\cdot)$, such that for $x \in D(\gamma)$

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Definition:(Excess distribution)

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