Application of regular variation

Example 1: Let X_1 and X_2 be two nonnegative i.i.d. r.v. with distribution function F, $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$. Let X_1 (X_2) represent the loss of a portfolio which consists of 1 unit of asset A_1 (A_2).

Assumption: The prices of A_1 and A_2 are identical and their logreturns are i.i.d..

Consider a portfolio P_1 containing 2 units of asset A_1 and a portfolio P_2 containing one unit of A_1 and one unit of A_2. Let L_i represent the loss of portfolio P_i, $i = 1, 2$.

Compare the probabilities of high losses in the two portfolios by computing the limit

$$\lim_{l \to \infty} \frac{\mathrm{Prob}(L_2 > l)}{\mathrm{Prob}(L_1 > l)}.$$

Application of regular variation (contd.)

Example 2: Let $X, Y \geq 0$ be two r.v. which represent the losses of two business lines of an insurance company (e.g. fire and car accidents).

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Assumptions

- $\bar{F} \in RV_{-\alpha}$, for some $\alpha > 0$, where F is the distribution function of X.
- $E(Y^k) < \infty$, $\forall k > 0$.

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- $E(Y^k) < \infty$, $\forall k > 0$.

Compute $\lim_{x \to \infty} P(X > x | X + Y > x)$.

Classical extreme value theory

Let (X_k), $k \in \mathbb{N}$, be non-degenerate i.i.d. r.v. with distribution function F. For $n \geq 1$ define $S_n := \sum_{i=1}^{n} X_i$ and $M_n := \max\{X_i : 1 \leq i \leq n\}$

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Consider first the limit distribution of S_n.

Question: What kind of non-degenerate r.v. Z are the limit distributions of $a_n^{-1}(S_n - b_n)$, for some sequences of reals $a_n > 0$ und $b_n \in \mathbb{R}$, $n \in \mathbb{N}$?

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Definition: A r.v. X is called *stable, (α-stable, Lévy-stable)*, iff for all $c_1, c_2 \in \mathbb{R}_+$ and the i.i.d. copies X_1 and X_2 of X, there exist constantes $a(c_1, c_2) \in \mathbb{R}$ and $b(c_1, c_2) \in \mathbb{R}$, such that $c_1X_1 + c_2X_2$ und $a(c_1, c_2)X + b(c_1, c_2)$ are identically distributed.

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Theorem

The family of stable distributions coincides whith the limit distributions of appropriately normalized and centralized sums of i.i.d. r.v..

Proof e.g. in Rényi, 1962.

Stable distributions (contd.)

Theorem: The characteristic function of a stable distribution X is given as:

$$\varphi X(t) = E[\exp\{iXt\}] = \exp\{i\gamma t - c|t|^{\alpha}(1 + i\beta \mathrm{signum}(t) z(t, \alpha))\}, \quad (4)$$

where $\gamma \in \mathbb{R}$, $c > 0$, $\alpha \in (0, 2]$, $\beta \in [-1, 1]$ and

$$z(t, \alpha) = \begin{cases} \tan(\frac{\pi\alpha}{2}) & \text{wenn } \alpha \neq 1 \\ -\frac{2}{\pi} \ln|t| & \text{wenn } \alpha = 1 \end{cases}$$

Proof: Lévy 1954, Gnedenko und Kolmogoroff 1960.

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Theorem: The characteristic function of a stable distribution X is given as:

$$\varphi_{\mathrm{X}}(\mathrm{t}) = \mathrm{E}\left[\exp\{\mathrm{iXt}\}\right] = \exp\{\mathrm{i}\gamma\mathrm{t} - \mathrm{c}|\mathrm{t}|^{\alpha}(1 + \mathrm{i}\beta\mathrm{signum}(\mathrm{t})\mathrm{z}(\mathrm{t},\alpha))\}\,, (4)$$

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Definition: The parameter α in (4) is called *the form parameter or characteristical exponent*, the corresponding distribution is called α-stable and its distribution function is denoted by G_{α}.

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Definition: The parameter α in (4) is called *the form parameter or characteristical exponent*, the corresponding distribution is called α-stable and its distribution function is denoted by G_α.

Definition: Let X be a r.v. with distribution function F. Assume that there exists two sequences of reals $a_n > 0$ and $b_n \in \mathbb{R}$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} a_n^{-1}(S_n - b_n) = G_\alpha$, for some α-stable distribution G_α. Then we say that "F belongs to the domain of attraction of G_α".
Notation: $F \in DA(G_\alpha)$.

Stable distributions (contd.)

Remark: $X \sim G_2 \iff \varphi_X(t) = \exp\{i\gamma t - \frac{1}{2}t^2(2c)\} \iff X \sim N(\gamma, 2c)$

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Exercise: Show that $F \in DA(G_2) \Longleftrightarrow F \in DA(\phi)$, where ϕ is the standard normal distribution $N(0,1)$.
Hint: The Convergence to Types Theorem could be used.

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Hint: The Convergence to Types Theorem could be used.

Definition: The r.v. Z and $\tilde{Z}$ *are of the same type* if there exist the constants $\sigma > 0$ and $\mu \in \mathbb{R}$, such that $\tilde{Z} \stackrel{d}{=} (Z - \mu)/\sigma$, i.e. $\tilde{F}(x) = F(\mu + \sigma x)$, $\forall x \in \mathbb{R}$, where F and $\tilde{F}$ are the distribution functions of Z and $\tilde{Z}$, respecitvely.

The Convergence to Types Theorem

Let Z, $\tilde{Z}$, Y_n, $n \geq 1$, be two not almost surely constant r.v.
Let $a_n, \tilde{a}_n, b_n, \tilde{b}_n \in \mathbb{R}$, $n \in \mathbb{N}$, be sequences of reals with $a_n, \tilde{a}_n > 0$.

(i) If

$$\lim_{n\to\infty} a_n^{-1}(Y_n - b_n) = Z \text{ and } \lim_{n\to\infty} \tilde{a}_n^{-1}(Y_n - \tilde{b}_n) = \tilde{Z} \tag{5}$$

then there exist $A > 0$ und $B \in \mathbb{R}$, such that

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Proof: See Resnick 1987, Prop. 0.2, Seite 7.

Characterization of the domain of attraction

(i) Let ϕ be the standard normal distribution function. The equivalence

$$F \in \mathrm{DA}(\phi) \Longleftrightarrow \lim_{x \to \infty} \frac{x^2 \int_{[-x,x]^C} dF(y)}{\int_{[-x,x]} y^2 dF(y)} = 0$$

holds, where $[-x, x]^C$ is the complement of $[-x, x]$ in $\mathbb{R}$.

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holds, where L is a slowly varying function around infinity and $c_1, c_2 \geq 0$ with $c_1 + c_2 > 0$.

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Remark: Let $F \in \mathrm{DA}(G_\alpha)$ for $\alpha \in (0,2)$. Then $E(|X|^\delta) < \infty$ for $\delta < \alpha$ and $E(|X|^\delta) = \infty$ for $\delta > \alpha$ hold.

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Proof: See Resnick 1987 (or a demanding homework!)

Limit distributions of normalized and centered maxima

Let (X_k), $k \in \mathbb{N}$, be non-gegenerate i.i.d. r.v. with distribution function F.

For $n \geq 1$, set $M_n := \max\{X_i : 1 \leq i \leq n\}$

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Consider $\lim_{n\to\infty} P\left(a_n^{-1}(M_n - b_n) \leq x\right) = \lim_{n\to\infty} P\left(M_n \leq u_n\right)$, where $u_n = a_n x + b_n$, $\forall n \in \mathbb{N}$.

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Theorem: (Poisson Approximation)
Let $\tau \in [0, \infty]$ and a sequence of reals $u_n \in \mathbb{R}$. Then the following holds

$$\lim_{n\to\infty} n\bar{F}(u_n) = \tau \iff \lim_{n\to\infty} P(M_n \leq u_n) = \exp\{-\tau\}.$$

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Exercise:
Use the convergence to types theorem to convince yourself that H and $\tilde{H}$ are of the same type, if
$\lim_{n\to\infty} a_n^{-1}(M_n - b_n) = H$ and $\lim_{n\to\infty} \tilde{a}_n^{-1}(M_n - \tilde{b}_n) = \tilde{H}$.

Max-stable distributions and the Fischer-Tippet theorem

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Definition: A non-degenarate r.v. X is called *max-stable* iff for any $n \geq 2$ max$\{X_1, X_2, \ldots, X_n\} \stackrel{d}{=} a_n X + b_n$ for indepedent copies X_1,X_2, $\ldots$,X_n of X and appropriate constants $a_n > 0$ and $b_n \in \mathbb{R}$.

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Theorem: (Proof in McNeil, Frey und Embrechts, 2005.)
The class of max-stable distributions coincides with the class of non-degenerate limit distributions of normalized and centered maxima of i.i.d. r.v.

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Theorem: (Fischer-Tippet Theorem, Proof in Resnick 1987, page 9-11)
Let (X_k) be a sequence of i.i.d. r.v.. If the constants $a_n, b_n \in \mathbb{R}$, $a_n > 0$, and a non-degenerate disribution H exist, such that $\lim_{n\to\infty} a_n^{-1}(M_n - b_n) = H$, then H is of the same type as one of the following three distributions:

$$\begin{array}{lrl}
\text{Fréchet} & \Phi_\alpha(x) = \begin{cases} 0 & x \leq 0 \\ \exp\{-x^{-\alpha}\} & x > 0 \end{cases} & \alpha > 0 \\
\text{Weibull} & \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & x \leq 0 \\ 1 & x > 0 \end{cases} & \alpha > 0 \\
\text{Gumbel} & \Lambda(x) = \exp\{-e^{-x}\} & x \in \mathbb{R}
\end{array}$$

Extreme value distributions

The distributions Φ_α, Ψ_α and Λ are called *standard extreme value distributions* (standard evd). The distributions which are of the same type as the standard evt are called *extreme value distributions* (evd).

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Definition: We say that the r.v. X (or the corresponding distribuion) belongs to the *maximum domain of attraction* of the evd H iff there exist constants $\mathrm{a_n} > 0$ and $\mathrm{b_n} \in \mathbb{R}$ such that $\lim_{\mathrm{n}\to\infty} \mathrm{a_n^{-1}}(\mathrm{M_n} - \mathrm{b_n}) = \mathrm{H}$ holds. Notation: $\mathrm{X} \in \mathrm{MDA}(\mathrm{H})$ ($\mathrm{F} \in \mathrm{MDA}(\mathrm{H})$).

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Theorem: (Characterisation of MDA)
$F \in MDA(H)$ with normalizing and centering constants $a_n > 0$ snd $b_n \in \mathbb{R}$ holds, iff

$$\lim_{n\to\infty} n\bar{F}(a_n x + b_n) = -\ln H(x), \forall x \in \mathbb{R},$$

where $-\ln H(x)$ is replaced by ∞ if $H(x) = 0$.

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$$\lim_{n\to\infty} n\bar{F}(a_n x + b_n) = -\ln H(x), \forall x \in \mathbb{R},$$

where $-\ln H(x)$ is replaced by ∞ if $H(x) = 0$.

Hint for the proof: apply the theorem about the Poisson approximation.

Extreme value distributions

The distributions Φ_α, Ψ_α and Λ are called *standard extreme value distributions* (standard evd). The distributions which are of the same type as the standard evt are called *extreme value distributions* (evd).

Definition: We say that the r.v. X (or the corresponding distribuion) belongs to the *maximum domain of attraction* of the evd H iff there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n^{-1}(M_n - b_n) = H$ holds. Notation: $X \in \mathrm{MDA}(H)$ ($F \in \mathrm{MDA}(H)$).

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Hint for the proof: apply the theorem about the Poisson approximation.

There exist distributions which do not belong to the MDA of any evd !
Example: (The Poisson distribution)
Let $X \sim P(\lambda)$, i.e. $P(X = k) = e^{-\lambda}\lambda^k/k!$, $k \in \mathbb{N}_0$, $\lambda > 0$. Show that there exist no evd Z such that $X \in \mathrm{MDA}(Z)$.

Examples and the generalized evd

Example: (Maxima of the exponential distribution)
Let (X_k) be a sequence of i.i.d. r.v. with distribution function F, $F(x) = 1 - e^{-x}$ for $x \geq 0$. Show that $F \in \mathrm{MDA}(\Lambda)$ with normalizing and centering constants $a_n = 1$ and $b_n = \ln n$.

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Example: (Maxima of the Cauchy distribution)
Let (X_k) be a sequence of i.i.d. r.v. with distribution function F and density function f, $f(x) = (\pi(1 + x^2))^{-1}$ for $x \in \mathbb{R}$. Show that $F \in \mathrm{MDA}(\Phi_1)$ with normalizing and centering constants $a_n = n/\pi$ and $b_n = 0$.

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Definition: (The generalized extreme value distribution (gevd))
Let the distribution function H_γ be given as follows:

$$H_\gamma(x) = \begin{cases} \exp\{-(1+\gamma x)^{-1/\gamma}\} & \text{wenn } \gamma \neq 0 \\ \exp\{-\exp\{-x\}\} & \text{wenn } \gamma = 0 \end{cases}$$

where $1 + \gamma x > 0$, i.e. the definition area of H_γ is given as

$$\begin{array}{ll} x > -\gamma^{-1} & \text{wenn } \gamma > 0 \\ x < -\gamma^{-1} & \text{wenn } \gamma < 0 \\ x \in \mathbb{R} & \text{wenn } \gamma = 0 \end{array}$$

H_γ is called *generalized extreme value distribution (gevd)*.