Risk and Management: Goals and Perspective

Etymology: Risicare

Risk (Oxford English Dictionary): (Exposure to) the possibility of loss, injury, or other adverse or unwelcome circumstance; a chance or situation involving such a possibility.
Finance: The possibility that an actual return on an investment will be lower than the expected return.

Risk management: is the identification, assessment, and prioritization of risks followed by coordinated and economical application of resources to minimize, monitor, and control the probability and/or impact of unfortunate events or to maximize the realization of opportunities.
Risk management's objective is to assure uncertainty does not deflect the endeavor from the business goals.

Risk and Management: Goals and Perspective

Subject of risk managment:

- Identification of risk sources (determination of exposure)
- Assessment of risk dependencies
- Measurement of risk
- Handling with risk
- Control and supervision of risk
- Monitoring and early detection of risk
- Development of a well structured risk management system

Risk and Management: Goals and Perspective

Main questions addressed by strategic risk managment:

- Which are the strategic risks?
- Which risks should be carried by the company?
- Which instruments should be used to control risk?
- What resources are needed to cover for risk?
- What are the risk adjusted measures of success used as steering mechanisms?

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Examples: standard deviation, quantile of the loss distribution, ...

Types of risk

For an organization risk arises through events or activities which could prevent the organization from fulfilling its goals and executing its strategies.
Financial risk:

- Market risk
- Credit risk
- Operational risk
- Liquidity risk, legal (judicial) risk, reputational risk

The goal is to estimate these risks as precisely as possible, ideally based on the loss distribution (LD).

Regulation and supervision

1974: Establishment of Basel Committee on Banking Supervision (BCBS).
Risk capital depending on GD/LD.
Suggestions and guidelines on the requirements and methods used to *compute the risk capital*. Aims at *internationally accepted standards* for the computation of the risk capital and *statutory dispositions* based on those standards.
Control by the supervision agency.

1988 Basel I: International minimum capital requirements especially with respect to (w.r.t.) credit risk.

1996 Standardised models are formulated for the assessment of market risk with an option to use value at risk (VaR) models in larger banks

2007 Basel II: minimum capital requirements w.r.t. credit risk, market risk and operational risk, procedure of control by supervision agencies, market discipline[1].

2010 BASEL III - Improvement and further development of BASEL II w.r.t. applicability, operational riskr und liquidity risk

[1] see http://www.bis.org

Assessment of the loss function

Loss operators

$V(t)$ - Value of portfolio at time t

Time unit Δt

Loss in time interval $[t, t+\Delta t]$: $L_{[t,t+\Delta t]} := -(V(t+\Delta t) - V(t))$

Discretisation of time: $t_n := n\Delta t,\ n = 0, 1, 2, \ldots$

$$L_{n+1} := L_{[t_n, t_{n+1}]} = -(V_{n+1} - V_n), \text{ where } V_n := V(n\Delta t)$$

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Let $Z_{n,i} := \ln S_{n,i}$, $X_{n+1,i} := \ln S_{n+1,i} - \ln S_{n,i}$
Let $w_{n,i} := \alpha_i S_{n,i}/V_n$, $i = 1, 2, \ldots, d$, be the relative portfolio weights.

Loss operator of an asset portfolio (cont.)

The following holds:

$$L_{n+1} := -\sum_{i=1}^{d} \alpha_i S_{n,i} \left(\exp\{X_{n+1,i}\} - 1 \right) =$$

$$-V_n \sum_{i=1}^{d} w_{n,i} \left(\exp\{X_{n+1,i}\} - 1 \right) =: l_n(X_{n+1})$$

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Linearisation $e^x = 1 + x + o(x^2) \sim 1 + x$ implies

$$L_{n+1}^{\Delta} = -V_n \sum_{i=1}^{d} w_{n,i} X_{n+1,i} =: l_n^{\Delta}(X_{n+1}),$$

where L_{n+1} ($L_{n+1]}^{\Delta}$) is the (linearised) loss function and l_n (l_n^{Δ}) is the (linearised) loss operator.

The general case

Let $V_n = f(t_n, Z_n)$ and $Z_n = (Z_{n,1}, \ldots, Z_{n,d})$, where Z_n is a vector of risk factors

Risk factor changes: $X_{n+1} := Z_{n+1} - Z_n$

$$L_{n+1} = -\Big(f(t_{n+1}, Z_n + X_{n+1}) - f(t_n, Z_n) \Big) =: l_n(X_{n+1}), \text{ where}$$

$$l_n(x) := -\Big(f(t_{n+1}, Z_n + x) - f(t_n, Z_n) \Big) \text{ is the loss operator.}$$

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The linearised loss:

$$L^{\Delta}_{n+1} = -\Big(f_t(t_n, Z_n)\Delta t + \textstyle\sum_{i=1}^{d} f_{z_i}(t_n, Z_n) X_{n+1,i} \Big),$$

where f_t and f_{z_i} are the partial derivatives of f.

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Definition: An *European call option (ECO)* on a certain asset S grants its holder the right but not the obligation to buy asset S at a specified day T (*execution day*) and at a specified price K (*strike price*). The option is bought by the owner at a certain price at day 0.

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Definition: A currency forward or an FX forward (FXF) is a contract between two parties to buy/sell an amount $\bar{V}$ of foreign currency at a future time T for a specified exchange rate $\bar{e}$. The party who is going to buy the foreign currency is said to hold a long position and the party who will sell holds a short position.

Example A bond portfolio

Let $B(t, T)$ be the price of the ZCB with maturity T at time $t < T$.

The *continuously compounded yield*, $y(t, T) := -\frac{1}{T-t} \ln B(t, T)$, would represent the continuous interest rate which was dealt with at time t as being constant for the whole interval $[t, T]$.

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Portfolio value at time t_n:
$V_n = \sum_{i=1}^{d} \alpha_i B(t_n, T_i) = \sum_{i=1}^{d} \alpha_i exp\{-(T_i - t_n) Z_{n,i}\} = f(t_n, Z_n)$
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A bond portfolio (contd.)

$$l_{[n]}(x) = -\sum_{i=1}^{d} \alpha_i B(t_n, T_i) \left(exp\{Z_{n,i}\Delta t - (T_i - t_{n+1})x_i\} - 1\right)$$

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<u>A long position</u> over $(\bar{V})$ units of a <u>FX forward</u> with maturity T

$$\Longleftrightarrow$$

<u>a long position</u> over $\bar{V}$ units of a <u>foreign zero-coupon</u> bond (ZCB) with maturity T and <u>a short position</u> over $\bar{e}\bar{V}$ units of a <u>domestic zero-coupon bond</u> with maturity T.

A currency forward portfolio (contd.)

Assumptions:

Euro investor holds a long position of a USD/EUR forward over $\bar{V}$ USD.

Let $B^f(t, T)$ ($B^d(t, T)$) be the price of a USD based (EUR-based) ZCB.

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Risk factors: $Z_n = (\ln e(t_n), y^f(t_n, T))^T$
Value of the long position (in Euro): $V_n = \bar{V} \exp\{Z_{n,1} - (T - t_n)Z_{n,2}\}$

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The linearized loss: $L^{\Delta}_{n+1} = -V_n(Z_{n,2}\Delta t + X_{n+1,1} - (T - t_{n+1})X_{n+1,2})$
where $X_{n+1,1} := \ln e(t_{n+1}) - \ln e(t_n)$ und
$X_{n+1,2} := y^f(t_{n+1}, T) - y^f(t_n, T)$

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Consider an ECO over an asset S with *execution date* T, price S_T at time T and *strike price* K.

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The greeks: C_t - theta, C_S - delta, C_r - rho, C_σ - Vega

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  Disadvantages: no difference between long and short positions, diversification effects are not condidered

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  Portfolio risk:

  $\Psi[\chi, w] = \max\{w_1 l_{[n]}(X_1), w_2 l_{[n]}(X_2), \ldots, w_N l_{[n]}(X_N)\}$

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Scenarios 1 to 8		Scenarios 9 to 14	
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$\nearrow$	$\nearrow \frac{1}{3} * Range$	$\nearrow$	$\searrow \frac{1}{3} * Range$
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Scenarios i, $i = 15, 16$ represent an extreme increase or decrease of the future price, respectively. The weights are $w_i = 1$, for $i \in \{1, 2, \ldots, 14\}$, and $w_i = 0.35$, for $i \in \{15, 16\}$.

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An appropriate model (zB. Black-Scholes) is used to generate the option prices in the different scenarios.

- **Risk measures based on the loss distribution**

  Let $F_L := F_{L_{n+1}}$ be the loss distribution of L_{n+1}.

  The parameters of F_L will be estimated in terms of historical data, either directly or bin terms of risk factors.

  1. **The standard deviation** $std(L) := \sqrt{\sigma^2(F_L)}$
     It is used frequently in portfolio theory.

     Disadvantages:

     - STD exists only for distributions with $E(F_L^2) < \infty$, not applicable to leptocurtic ("fat tailed") loss distributions;
     - gains and losses equally influence the STD.

     Example
     $L_1 \sim N(0, 2)$, $L_2 \sim t_4$ (Student's distribution with $m = 4$ degrees of freedom)
     $\sigma^2(L_1) = 2$ and $\sigma^2(L_2) = \frac{m}{m-2} = 2$ hold
     However the probability of losses is much larger for L_2 than for L_1.
     Plot the logarithm of the quotient $\ln[P(L_2 > x)/P(L_1 > x)]$!

2. **Value at Risk** ($VaR_\alpha(L)$)

   Definition: Let L be the loss distribution and $\alpha \in (0,1)$ a given confindence level.

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$$VaR_\alpha(L) = \inf\{l \in \mathbb{R} \colon P(L > l) \leq 1 - \alpha\} =$$
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Definition: Let $F \colon A \to B$ be an increasing function. The function $F^{\leftarrow} \colon B \to A \cup \{-\infty, +\infty\}, y \mapsto \inf\{x \in \mathbb{R} \colon F(x) \geq y\}$ is called *generalized inverse function* of F.

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If F is strictly monotone increasing, then $F^{-1} = F^{\leftarrow}$ holds.

Exercise: Compute $F^{\leftarrow}$ for $F \colon [0, +\infty) \to [0, 1]$ with

$$F(x) = \begin{cases} 1/2 & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

Value at Risk (contd.)

Definition: Let $F\colon \mathbb{R} \to \mathbb{R}$ be a (monotone increasing) distribution function and $q_\alpha(F) := \inf\{x \in \mathbb{R}\colon F(x) \geq \alpha\}$ be α-quantile of F.

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Example: Let $L \sim N(\mu, \sigma^2)$.
Then $VaR_\alpha(L) = \mu + \sigma q_\alpha(\Phi) = \mu + \sigma \Phi^{-1}(\alpha)$ holds, where Φ is the distribution function of a random variable $X \sim N(0, 1)$.

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Exercise: Consider a portfolio consisting of 5 pieces of an asset A. The today's price of A is $S_0 = 100$. The daily logarithmic returns are i.i.d.: $X_1 = \ln \frac{S_1}{S_0}$, $X_2 = \ln \frac{S_2}{S_1}, \ldots \sim N(0, 0.01)$. Let L_1 be the 1-day portfolio loss in the time interval (today, tomorrow).

(a) Compute $VaR_{0.99}(L_1)$.

(b) Compute $VaR_{0.99}(L_{100})$ and $VaR_{0.99}(L_{100}^{\Delta})$, where L_{100} is the 100-day portfolio loss over a horizon of 100 days starting with today. L_{100}^{Δ} is the linearization of the above mentioned 100-day PF-portfolio loss.

Hint: For $Z \sim N(0,1)$ use the equality $F_Z^{-1}(0.99) \approx 2.3$.

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$CVaR_\alpha(L) = E(L|L \geq VaR_\alpha(L)) = \frac{E(LI_{[q_\alpha(L),\infty)}(L))}{P(L \geq q_\alpha(L))} =$
$\frac{1}{1-\alpha} E(LI_{[q_\alpha(L),\infty)}) = \frac{1}{1-\alpha} \int_{q_\alpha(L)}^{+\infty} l dF_L(l)$

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If F_L is discrete the *generalized CVaR* is defined as follows:

$$GCVaR_\alpha(L) := \frac{1}{1-\alpha} \left[E(LI_{[q_\alpha(L),\infty)}) + q_\alpha \left(1 - \alpha - P(L > q_\alpha(L)) \right) \right]$$

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Lemma Let α be a given confidence level and L a continuous loss function with distribution F_L.
Then $CVaR_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 VaR_p(L) dp$ holds.

Conditional Value at Risk (contd.)
Example 1:

(a) Let $L \sim Exp(\lambda)$. Compute $CVaR_\alpha(L)$.

(b) Let the distribution function F_L of the loss function L be given as follows : $F_L(x) = 1 - (1 + \gamma x)^{-1/\gamma}$ for $x \geq 0$ and $\gamma \in (0, 1)$. Compute $CVaR_\alpha(L)$.

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Example 2:

Let $L \sim N(0, 1)$. Let ϕ und Φ be the density and the distribution function of L, respectively. Show that $CVaR_\alpha(L) = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ holds.

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Exercise:

Let the loss L be distributed according to the Student's t-distribution with $\nu > 1$ degrees of freedom. The density of L is

$$g_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}$$

Show that $CVaR_\alpha(L) = \frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha} \left(\frac{\nu + (t_\nu^{-1}(a))^2}{\nu - 1}\right)$, where t_ν is the distribution function of L.