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Examples of finance instruments affected by credit risk

- bond portfolios
- OTC ("over the counter") transactions
- trades with credit derivatives
- ...

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L is a r.v. and its distribution depends from the c.d.f. of $(X_1, \ldots, X_n, \lambda_1, \ldots, \lambda_n)^T$ ab.

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Then we have $X_i = \begin{cases} 0 & S_i \neq 0 \\ 1 & S_i = 0 \end{cases}$

Models with latent variables (contd.)

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$S = (S_1, S_2, \ldots, S_n)^T$ is modelled by means of latent variables $Y = (Y_1, Y_2, \ldots, Y_n)^T$, e.g. Y_i could be the value of the assets of obligor i

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Let d_{ij}, $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, m+1$ be threshold values such that $d_{i,0} = -\infty$ und $d_{i,m+1} = \infty$ and $S_i = j \Longleftrightarrow Y_i \in (d_{i,j}, d_{i,j+1}]$.

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The probability that the fisrt k obligors default:

$$p_{1,2,\dots,k} := P(Y_1 \le d_{1,1}, Y_2 \le d_{2,1}, \dots, Y_k \le d_{k,1})$$

$$= C(F_1(d_{1,1}), F_2(d_{2,1}), \dots, F_k(d_{k,1}), 1, 1, \dots, 1) = C(p_1, p_2, \dots, p_k, 1, \dots, 1)$$

Thus the totalt defalut probability depends essentially on the copula C of $(Y_1, Y_2, \dots, Y_n)$.

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Notations:

$V_{A,i}(T)$: value of assets of firm i at time point T

$K_i := K_i(T)$: value of the debt of firm i at time point T

$V_{E,i}(T)$: value of equity of firm i at time point T

Assumption: future asset value is modelled by a geometric Brownian motion

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DD_i is called *distance-to-default*.

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This can be seen as a call option on the firms assets with a strike price equal to the book value of the firms liabilities.

Thus $V_{E,i}(T) = \max\{V_{A,i}(T) - K_i, 0\}$.

The Black-Scholes formula implies (option price theory):

$V_{E,i}(t) = C(V_{A,i}(t), r, \sigma_{A,i}) = V_{A,i}(t)\phi(e_1) - K_i e^{-r(T-t)}\phi(e_2)$, where

$e_1 = \frac{\ln(V_{A,i}(t) - \ln K_i + (r + \sigma_{A,i}^2/2)(T-t)}{\sigma_{A,i}(T-t)}$, $e_2 = e_1 - \sigma_{A,i}(T-t)$,

ϕ is the the standard normal distribution function and r is the risk free interest rate.

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$DD_i = \frac{\ln V_{A,i}(t) - \ln K_i + (\mu_{A,i} - \frac{\sigma_{A,i}^2}{2})(T-t)}{\sigma_{A,i}\sqrt{T-t}}$.

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Then $P(V_{A,i}(T) < K_i) = P(Y_i < -DD_i)$ and in the general setup of the latent variable model with $m = 1$ we have $d_{i1} = -DD_i$.

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Summary of the univariate KMV model to compute the default probability of a company:

- Estimate the asset value $V_{A,i}$ and the volatilty $\sigma_{A,i}$ by using observations of the market value and the volatility of equity $V_{E,i}$, $\sigma_{E,i}$, the book of liabilities K_i, and by solving the system of equations above.
- Compute the distance-to-default DD_i by means of the corresponding formula.
- Estimate the default probability p_i in terms of the empirical distribution which relates the distance to default with the expected default frequency.

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$$V_{A,i}(t) \exp \left\{ \left(\mu_{A,i} - \frac{\sigma_{A,i}^2}{2} \right) (T - t) + \sum_{j=1}^{m} \sigma_{A,i,j} \Bigg(W_j(T) - W_j(t) \Bigg) \right\},$$

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We get $V_{A,i}(T) < K_i \iff Y_i < -DD_i$ with

$$DD_i = \frac{\ln V_{A,i}(t) - \ln K_i + \left(\frac{-\sigma_{A,i}^2}{2} + \mu_{A,i} \right)(T-t)}{\sigma_{A,i}\sqrt{T-t}}.$$

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The probability that the k first firms default:

$$P(X_1 = 1, X_2 = 1, \ldots, X_k = 1) = P(Y_1 < -DD_1, \ldots, Y_k < -DD_k)$$
$$= C_{\Sigma}^{Ga}(\phi(-DD_1), \ldots, \phi(-DD_k), 1, \ldots, 1),$$

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Joint default frequency:

$JDF_{1,2,\ldots,k} = C_\Sigma^{Ga}(EDF_1, EDF_2, \ldots, EDF_k, 1, \ldots, 1)$,
where EDF_i is the default frequency for firm i, $i = 1, 2, \ldots, k$.