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Let $\phi(t) = 1 - t$, $t \in [0, 1]$. Then $\phi^{[-1]}(t) = \max\{1 - t, 0\}$ and

$$C_\phi(u_1, u_2) := \phi^{[-1]}(\phi(u_1) + \phi(u_2)) = \max\{u_1 + u_2 - 1, 0\} = W(u_1, u_2).$$

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Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and an Archimedian copula C generated by ϕ . Then

$$\rho_\tau(X_1, X_2) = 1 + 4 \int_0^1 \frac{\phi(t)}{\phi'(t)} dt \text{ holds.}$$

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See Nelsen 1999 for a proof.

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Multivariate Archimedian copulas

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Definition: A function $g: [0, \infty) \rightarrow [0, \infty)$ is called completely monotone iff all higher order derivatives of g exist and the following inequalities hold for $k \in \mathbb{N}_*$: $(-1)^k \left(\frac{d^k}{ds^k} g(s) \right) \Big|_{s=t} \geq 0, \forall t \in (0, \infty).$

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Theorem: (Kimberling 1974)

Let $\phi: [0, 1] \rightarrow [0, \infty]$ be a continuous, strictly monotone decreasing function with $\phi(0) = \infty$ and $\phi(1) = 0$. The function $C: [0, 1]^d \rightarrow [0, 1]$, $C(u) := \phi^{-1}(\phi(u_1) + \phi(u_2) + \dots + \phi(u_d))$ is a copula for $d \geq 2$ iff ϕ^{-1} is completely monotone on $[0, \infty)$.

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Lemma: A function $\psi: [0, \infty) \rightarrow [0, \infty)$ is completely monotone with $\psi(0) = 1$ iff ψ is the Laplace-Stieltjes transform of some distribution function G on $[0, \infty)$, i.e. $\psi(s) = \int_0^\infty e^{-sx} dG(x), s \geq 0$.

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Theorem: Let G be a distribution function on $[0, \infty)$ such that $G(0) = 0$. Let ψ be the Laplace-Stieltjes transform of G , i.e.

$\psi(s) = \int_0^\infty e^{-sx} dG(x)$ ψ for $s \geq 0$. Let X be a r.v. with distribution function G and let U_1, U_2, \dots, U_d be conditionally independent r.v. for $X = x$, $x \in \mathbb{R}^+$, with conditional distribution function

$$F_{U_k|X=x}(u) = \exp(-x\psi^{-1}(u)) \text{ for } u \in [0, 1].$$

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- ▶ have a closed form representation
- ▶ depend on a small number of parameters in general
- ▶ the generator function needs to fulfill quite restrictive technical assumptions

Simulation of Gaussian copulas

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Observe: Consider a symmetric positive definite matrix $R \in \mathbb{R}^{d \times d}$ and its Cholesky factorization $AA^T = R$ with $A \in \mathbb{R}^{d \times d}$. If $Z_1, Z_2, \dots, Z_d \sim N(0, 1)$ are independent, then $\mu + AZ \sim N_d(\mu, R)$.

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Algorithm: for the generation of a random vector $U = (U_1, U_2, \dots, U_d)$ whose distribution function is the copula C_R^{Ga} , R positive definite.

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- ▶ Output $U = (U_1, U_2, \dots, U_d)$; U has distribution function C_R^{Ga} .

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Algorithm: for the generation of a random vector $U = (U_1, U_2, \dots, U_d)$ whose distribution function is the copula $C_{\nu, R}^t$, R positive definite, $\nu \in \mathbb{N}$.

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- ▶ Output $U = (U_1, U_2, \dots, U_d)$; $U = (U_1, U_2, \dots, U_d)$ has distribution function $C_{\nu, R}^t$.

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Simulation of Archimedian copulas

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Input: The dimension $d \in \mathbb{N}$, the Archimedean Copula

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The generator $\varphi(t) = (t^{-\theta} - 1)/\theta$, $\theta > 0$ yields the Clayton copula C_θ^{Cl} .

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For $X \sim Gamma(1/\theta, 1)$ with d.f. $f_X(x) = x^{1/\theta-1} e^{-x}/\Gamma(1/\theta)$ we have:
 $E(e^{-sX}) = \int_0^\infty e^{-sx} \frac{1}{\Gamma(1/\theta)} x^{1/\theta-1} e^{-x} dx = (s+1)^{-1/\theta} = \tilde{\varphi}^{-1}(s)$.

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Set $(Z_1, Z_2)^T := (VS^\theta, (1 - V)S^\theta)$.

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The distribution function of $(\bar{F}(Z_1), \bar{F}(Z_2))^T$ is C_θ^{Gu} . Convince yourself!

Simulation of the Gumbel copula ($\theta \geq 1$) (contd.)

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Algorithm to generate a random vector $U = (U_1, U_2, \dots, U_d)$ with the Gumbel copula C_θ^{Gu} as distribution function.

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- ▶ Simulate two i.i.d. r.v. $V_1, V_2 \sim U(0, 1)$.
- ▶ Simulate two independent r.v. W_1, W_2 with $W_1 \sim \Gamma(1, 1)$, $W_2 \sim \Gamma(2, 1)$
- ▶ Set $S := I_{V_2 \leq 1/\theta} W_1 + I_{V_2 > 1/\theta} W_2$.
- ▶ Set $(Z_1, Z_2) := (V_1 S^\theta, (1 - V_1) S^\theta)$.
- ▶ The distribution function of $U = \left(\exp(-Z_1^{1/\theta}), \exp(-Z_2^{1/\theta}) \right)^T$ is C_θ^{Gu} .

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