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The proof uses the equality of Höffding:
Lemma: (The Höffding equality)
Let $(X_1, X_2)^T$ be a random vector with c.d.f. F and marginal d.f. F_1, F_2. If $cov(X_1, X_2) < \infty$ then the following equality holds:

$$cov(X_1, X_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (F(x_1, x_2) - F_1(x_1)F_2(x_2)) dx_1 dx_2 \,.$$

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Proof in McNeil et al., 2005.

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Hint: Observe that $X_1 \stackrel{d}{=} \exp(Z)$ and $X_2 \stackrel{d}{=} \exp(\sigma Z) \stackrel{d}{=} \exp(-\sigma Z)$. Moreover e^Z, $e^{\sigma Z}$ are co-monotone and e^Z,$e^{\sigma Z}$ are anti-monotone.

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Example: Determine two random vectors $(X_1,X_2)^T$ and $(Y_1,Y_2)^T$ with different c.d.f.s such that $F^{\leftarrow}_{X_1+X_2}(\alpha) \neq F^{\leftarrow}_{Y_1+Y_2}(\alpha)$ holds while $X_1, X_2, Y_1, Y_2 \sim N(0,1)$ and $\rho_L(X_1,X_2) = 0$, $\rho_L(Y_1,Y_2) = 0$ also hold.

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If $(X_1,X_2)^T$, $(Y_1,Y_2)^T$ represent the asset returns of two different portfolios consisting of two assets each, then we have two portfolios with the same marginal distributions of their assets and the same linear correlation coefficient, respectively, but having different value at risk.

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Conclusion: The marginal distributions of the assets in a portfolio and the linear correlation between the assets do not determine the loss distribution, in particular, they do not determine the risk measure of the portfolio.

The rang correlation Kendall's Tau

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Let $(x, y)^T$ and $(\tilde{x}, \tilde{y})^T$ be two samples of a random vector $(X, Y)^T$. $(x, y)^T$ und $(\tilde{x}, \tilde{y})^T$ are called *concordant* if $(x - \tilde{x})(y - \tilde{y}) > 0$ and *discordant* if $(x - \tilde{x})(y - \tilde{y}) < 0$.

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Definition: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions. The Kendall's Tau of $(X_1, X_2)^T$ is defined as
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Equivalently: $\rho_\tau(X_1, X_2) = E(sign[(X_1 - X_1')(X_2 - X_2')])$. In the d-dimensional case $X \in \mathbb{R}^d$: $\rho_\tau(X) = cov(sign(X - X'))$, where $X' \in \mathbb{R}^D$ is an independent copy of $X \in \mathbb{R}^d$.

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The sample Kendall's Tau:

Let $\{(x_1, y_1)^T, (x_2, y_2)^T, \ldots, (x_n, y_n)^T\}$ be a sample of size n of the random vector $(X, Y)^T$ with continuous marginal distributions. Let c be the number concordant pairs in the sample and let d be the number of discordant pairs. Then the sample Kendall's Tau is given as

$$\tilde{\rho}_\tau(X, Y) = \frac{c - d}{c + d} \overset{a.s.}{=} \frac{c - d}{n(n-1)/2}$$

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Equivalent definition (without a proof):
Let F_1 und F_2 be the continuous marginal distributions of $(X_1, X_2)^T$.
Then $\rho_S(X_1, X_2) = \rho_L(F_1(X_1), F_2(X_2))$ holds, i.e. the Spearman's Rho is the linear correlation of the unique copula of $(X_1, X_2)^T$.

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In the d-dimensional case $X \in \mathbb{R}^d$:
$\rho_S(X) = \rho(F_1(X_1), F_2(X_2), \ldots, F_d(X_d))$ is the correlation matrix of the unique copula of X, where F_1,F_2,...,F_d are the continuous marginal distributions of X.

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- Let F_1, F_2 be the continuous marginal distributions of $(X_1, X_2)^T$ and let T_1, T_2 be strictly monotone functions on $[-\infty, \infty]$. Then the following equalties hold $\rho_\tau(X_1, X_2) = \rho_\tau(T_1(X_1), T_2(X_2))$ and $\rho_S(X_1, X_2) = \rho_S(T_1(X_1), T_2(X_2))$.

(See Embrechts et al., 2002).

Tail dependence coefficients

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Definition: Let $(X_1, X_2)^T$ be a random vector with marginal distributions F_1 und F_2.

The coefficent $\lambda_U(X_1, X_2)$ *of the upper tail dependency* of $(X_1, X_2)^T$ is defined as $\lambda_U(X_1, X_2) = \lim_{u \to 1^-} P(X_2 > F_2^{\leftarrow}(u) | X_1 > F_1^{\leftarrow}(u))$, provided that the limit exists.

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The coefficent $\lambda_L(X_1, X_2)$ of the lower tail dependency of $(X_1, X_2)^T$ is defined as $\lambda_L(X_1, X_2) = \lim_{u \to 0^+} P(X_2 \leq F_2^{\leftarrow}(u) | X_1 \leq F_1^{\leftarrow}(u))$ provided that the limit exists.

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If the limit exists and $\lambda_U > 0$ ($\lambda_L > 0$) we say that $(X_1, X_2)^T$ have an upper (a lower) tail dependence.

Tail dependency and survival copulas

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Definition: Let the copula C be the c.d.f. of a random vector $(U_1, U_2, \ldots, U_d)$ with $U_i \sim U[0,1]$, $i = 1, 2, \ldots, d$. The c.d.f. of $(1 - U_1, 1 - U_2, \ldots, 1 - U_d)$ is called *survival copula* of C and is denoted by $\hat{C}$.

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Lemma: Let X be a random vector with multivariate tail distribution function $\bar{F}$ ($\bar{F}(x_1, x_2, \ldots, x_d) := Prob(X_1 > x_1, X_2 > x_2, \ldots, X_d > x_d)$) and marginal distributions F_i, $i = 1, 2, \ldots, d$. Let $\bar{F}_i := 1 - F_i$, $i = 1, 2, \ldots, d$. Then the following holds

$$\bar{F}(x_1, x_2, \ldots, x_d) = \hat{C}(\bar{F}_1(x_1), \bar{F}_2(x_2), \ldots, \bar{F}_d(x_d)).$$

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Lemma: For any copula C and its survival copula $\hat{C}$ the following holds $\hat{C}(1 - u_1, 1 - u_2) = 1 - u_1 - u_2 + C(u_1, u_2)$.

Tail dependency and survival copulas

Definition: Let the copula C be the c.d.f. of a random vector $(U_1, U_2, \ldots, U_d)$ with $U_i \sim U[0,1]$, $i = 1, 2, \ldots, d$. The c.d.f. of $(1-U_1, 1-U_2, \ldots, 1-U_d)$ is called *survival copula* of C and is denoted by $\hat{C}$.

Lemma: Let X be a random vector with multivariate tail distribution function $\bar{F}$ ($\bar{F}(x_1, x_2, \ldots, x_d) := Prob(X_1 > x_1, X_2 > x_2, \ldots, X_d > x_d)$) and marginal distributions F_i, $i = 1, 2, \ldots, d$. Let $\bar{F}_i := 1 - F_i$, $i = 1, 2, \ldots, d$. Then the following holds

$$\bar{F}(x_1, x_2, \ldots, x_d) = \hat{C}(\bar{F}_1(x_1), \bar{F}_2(x_2), \ldots, \bar{F}_d(x_d)).$$

Lemma: For any copula C and its survival copula $\hat{C}$ the following holds $\hat{C}(1-u_1, 1-u_2) = 1 - u_1 - u_2 + C(u_1, u_2)$.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and a unique copula C. The following equalities hold $\lambda_U(X_1, X_2) = \lim_{u \to 1^-} \frac{1-2u+C(u,u)}{1-u}$ and $\lambda_L(X_1, X_2) = \lim_{u \to 0^+} \frac{C(u,u)}{u}$, provided that the limits exist.

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$$C_\theta^{\mathrm{Gu}}(u_1, u_2) = \exp\left(-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta\right]^{1/\theta}\right), \ \theta \geq 1$$

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The Clayton family of copulas:

$$C_\theta^{\mathrm{Cl}}(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{1/\theta}, \ \theta > 0$$

We have $\lambda_U = 0$, $\lambda_L = 2^{-1/\theta}$.

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Definition: Let X be a d-dimensional random vector. Let $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d\times d}$ be constants, and let $\psi\colon [0,\infty) \to \mathbb{R}$ be a function such that $\phi_{X-\mu} = \psi(t^T\Sigma t)$ holds for the characteristic function $\phi_{X-\mu}$ of $X-\mu$. Then X is an elliptically distributed random vector with parameters μ, Σ, ψ. Notation: $X \sim E_d(\mu,\Sigma,\psi)$.

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Theorem:(Stochastic representation)
A d-dimensional random vector X is elliptically distributed, $X \sim E_d(\mu,\Sigma,\psi)$ with *rang*$(\Sigma)=k$, iff there exist a matrix $A \in \mathbb{R}^{d\times k}$, $A^TA=\Sigma$, a nonnegative r.v. R and a k-dimensional random vector U unformly distributed on the unit ball $\mathcal{S}^{k-1} = \{z \in \mathbb{R}^k\colon z^Tz=1\}$, such that R and U are independent and $X \stackrel{d}{=} \mu + RAU$.

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Remark: An elliptically distributed random vector X ist *radial symmetric*, i.e. $X - \mu \stackrel{d}{=} \mu - X$.