Risk theory and risk management in actuarial science
Winter term 2019/2020

5th work sheet

31. The Gumbel family C_θ^{Gu} and the Clayton family C_θ^{Cl} are two one-parametric families of copulas given as

$$C_\theta^{\mathrm{Gu}}(u_1,u_2) := \exp\left(-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta\right]^{1/\theta}\right),\ \theta \ge 1, \text{ and}$$

$$C_\theta^{\mathrm{Cl}}(u_1,u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta},\ \theta > 0.$$

   (a) Compute Kendall's tau ρ_τ as well as the coefficients λ_U, λ_L of the upper and lower tail dependence for the copulas C_θ^{Gu}, C_θ^{Cl}, respectively.

   (b) The independence copula Π is given by $\Pi(u_1,u_2) := u_1u_2$, for $(u_1,u_2) \in [0,1]^2$. Show that C_θ^{Gu} tends to the independence copula Π if θ tends to 1 and to the upper Fréchet bound M if θ tends to infinity. In this case we say that *the lower limit of the Gumbel copula is the independence copula* Π *and its upper limit is the Fréchet upper bound* M. Analogously show that the lower limit of the Clayton copula is the independence copula Π for $\theta \to 0^+$ and its upper limit is the Fréchet upper bound M for $\theta \to +\infty$. Now considerer an extension of the Clayton copula C_θ^{Cl} for $\theta \in [-1,0)$, defined as an Archimedian copula with generator $\phi_\theta(t) = \frac{1}{\theta}(t^{-\theta} - 1)$ for $t \in (0,1]$ and $\phi_\theta(0) = +\infty$. Show that for $\theta = -1$ the Clayton copula C_{-1}^{Cl} coincides with the Fréchet lower bound W.

32. (a) Let $(X_1,X_2)^T$ be a t-distributed random vektor with ν degrees of freedom, expected value $(0,0)$ and linear correlation coefficient matrix $\rho \in (-1,1]$, i.e. $(X_1,X_2)^T \sim t_2(\vec{0},\nu,R)$ where R is 2×2 matrix with 1 on the diagonal and ρ outside the diagonal. Show that the following equality holds for $\rho > -1$:

$$\lambda_U(X_1,X_2) = \lambda_L(X_1,X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)$$

   Hint: Use the fact (no need to prove it!) that conditional on $X_1 = x$ the following holds

$$\left(\frac{\nu+1}{\nu+x^2}\right)^{1/2} \frac{X_2 - \rho x}{\sqrt{1-\rho^2}} \sim t_{\nu+1}\,.$$

   Recall the stochatic representation of the bivariate t-distribution as $\mu + \sqrt{W}AZ$, where Z is bivariate standard normally distributed and W is such that $\frac{\nu}{W} \sim \chi^2_\nu$ while being independent on Z (cf. lecture).

   (b) Apply (a) to conclude that for a random vector with continuous marginal distributions $(X_1,X_2)^T$ and a t-copula $C_{\nu,R}^t$ with ν degrees of freedom and a correlation matrix R as in (a) the following equalities holds:

$$\lambda_U(X_1,X_2) = \lambda_L(X_1,X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\right)\,.$$

33. **Archimedian Copulas**

   (a) Show that for every $\theta \in \mathbb{R}\setminus\{0\}$ the function $\phi_\theta^{Fr}(t) = -\ln\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right)$ generates an Archmedian copula, the so-called Frank copula $C_\theta^{Fr}\colon [0,1]^2 \to [0,1]$. Check that the following equality holds $\forall u_1,u_2 \in [0,1]$:

$$C_\theta^{Fr}(u_1,u_2) = -\frac{1}{\theta}\ln\left(1 + \frac{(\exp(-\theta u_1) - 1)(\exp(-\theta u_2) - 1)}{\exp(-\theta) - 1}\right),\theta \in \mathbb{R}\setminus\{0\}\,.$$

(b) Show that for every $\theta > 0$ and for every $\delta \geq 1$ the function $\phi^{GC}_{\theta,\delta}(t) = \theta^{-\delta}(t^{-\theta} - 1)^{\delta}$ generates an Archmedian copula, the so-called *generalized Clayton copula* $C^{GC}_{\theta,\delta}\colon [0,1]^2 \to [0,1]$. Check that the following equality holds $\forall u_1, u_2 \in [0,1]$:

$$C^{GC}_{\theta,\delta}(u_1,u_2) = \{[(u_1^{-\theta} - 1)^{\delta} + (u_2^{-\theta} - 1)^{\delta}]^{1/\delta} + 1\}^{-1/\theta}, \, \theta \geq 0 \, , \delta \geq 1 \, .$$

(c) Compute Kendall's tau ρ_τ as well as the coefficients λ_U, λ_L of the upper and lower tail dependency for the copulas C^{Fr}_{θ} and $C^{GC}_{\theta,\delta}$, respectively.

34. **Asymmetric bivariate copulas**
Let C_θ be any exchangeable bivariate copula. Then a parametric family of asymmetric copulas $C_{\theta,\alpha,\beta}$ is obtained by setting

$$C_{\theta,\alpha,\beta}(u_1,u_2) = u_1^{1-\alpha} u_2^{1-\beta} C_\theta(u_1^{\alpha}, u_2^{\beta}) \, , \; 0 \leq u_1, u_2 \leq 1, \tag{1}$$

where $0 \leq \alpha, \beta \leq 1$. When C_θ is an Archimedian copula we refer to the copulas constructed by (1) as asymmetric bivariate Archimedian copulas.

Check that $C_{\theta,\alpha,\beta}$ defined as above is a copula by constructing a random vector with distribution function $C_{\theta,\alpha,\beta}$ and observing that its margins are standard uniform on $[0,1]$. Show that such a random vector (U_1,U_2) can be generated as follows:

(a) Generate a random pair (V_1,V_2) with distribution function C_θ.

(b) Generate, independently of V_1, V_2, two independent standard uniform variables $\bar{U}_1$ and $\bar{U}_2$.

(c) Set $U_1 = \max\left\{V_1^{1/\alpha}, \bar{U}_1^{1/(1-\alpha)}\right\}$ and $U_2 = \max\left\{V_2^{1/\beta}, \bar{U}_2^{1/(1-\beta)}\right\}$

Show that $C_{\theta,\alpha,\beta}$ is not exchangeable in general. What conditions should fulfill its parameters so as to obtain an exchangeable $C_{\theta,\alpha,\beta}$? Which copula results from $C_{\theta,\alpha,\beta}$ if $\alpha = \beta = 0$? Which copula results from $C_{\theta,\alpha,\beta}$ if $\alpha = \beta = 1$?

35. **Three-dimensional non-exchangeable Archimedian copulas**
Suppose that ϕ_1 and ϕ_2 are two *strict* generators of Archimedian copulas, i.e. ϕ_1 and ϕ_2 are completely monotonic decreasing functions for which the pseudo-inverse function coincides with the ordinary inverse function. Consider

$$C(u_1,u_2,u_3) = \phi_2^{-1}(\phi_2 \circ \phi_1^{-1}(\phi_1(u_1) + \phi_1(u_2)) + \phi_2(u_3)). \tag{2}$$

If ϕ_1^{-1}, ϕ_2^{-1} and $\phi_2 \circ \phi_1^{-1}$ are completely monotonic decreasing functions mapping $[0,\infty]$ to $[0,\infty]$, then C defined by (2) is a copula.

Let (U_1,U_2,U_3) be a random vector with distribution function C as defined by (2).

(a) Show that if $\phi_1 \neq \phi_2$, then only U_1 and U_2 are exchangeable, i.e. $(U_1,U_2,U_3) \stackrel{d}{=} (U_2,U_1,U_3)$, but no other swapping of subscritps is possible. Show moreoever that if $\phi_1 = \phi_2$, than C is an exchangeable copula.

(b) Show that all bivariate margins of C defined by (2) are themselves Archimedian copulas; the margins C_{13} (obtained by components 1 and 3) and C_{23} (obtained by components 2 and 3) have generator ϕ_2, whereas the margin C_{12} (obtained by components 1 and 2) has generator ϕ_1.

36. **Equivalent threshold models**
Let $X = (X_1, X_2, \ldots, X_m)'$ be an m-dimensional random vector and let $D \in \mathbb{R}^{m\times n}$ be a deterministic matrix with elements d_{ij} such that for every i, $1 \leq i \leq m$, the elements of the i-th row form a set of increasing thresholds satisfying $d_{i,1} < d_{i2} \ldots < d_{in}$. Introduce additionally $d_{i0} = -\infty$, $d_{i,n+1} = +\infty$ and set

$$S_i = j \Longleftrightarrow d_{ij} < X_i \leq d_{i,j+1} \, , \text{for } j \in \{0,\ldots,n\}, i \in \{1,\ldots,m\}.$$

Then (X,D) is said to define a *threshold model for the state vector* $S = (S_1,\ldots,S_m)'$. We refere to X as the *vector of critical variables* and denote its marginal distribution functions by $F_i(x) =$

$P(X_i \le x)$, for $i \in \{1, 2, \dots, m\}$. The i-th row of D contains the critical thresholds for firm i. By definition, default (corresponding to event $S_i = 0$) occurs iff $X_i \le d_{i1}$, thus the default probability of company i is given by $\bar{p}_i := F_i(d_{i1})$. Let Y_i be the default indicator of company i, i.e. $Y_i \in \{0, 1\}$ with $Y_i = 1$ iff company 1 defaults, hence $Prob(Y_i = 1) = \bar{p}_i$ and $Prob(Y_i = 0) = 1 - \bar{p}_i$, for $1 \le i \le m$. We denote by $\rho(Y_i, Y_j)$ the *default correlation* of two firms $i \ne j$; this quantity depends on $E(Y_i, Y_j)$ (how?) which in turn depends on the joint distribution of (X_i, X_j), and hence on the copula of $(X_1, X_2, \dots, X_m)'$. (Notice that in general the latter is not fully determined by the *asset correlation* $\rho(X_i, X_j)$.)

Two threshold models (X, D) and $(\tilde{X}, \tilde{D})$ for the state vectors S and $\tilde{S}$, respectively, are called equivalent, iff S and $\tilde{S}$ have the same probability distribution.

Show that two threshod models (X, D) and $(\tilde{X}, \tilde{D})$ with state vectors S and $\tilde{S}$, respectively, are equivalent if the following conditions hold:

(a) The marginal distributions of the random vectors S and $\tilde{S}$ coincide, i.e. $P(S_i = j) = P(\tilde{S}_i = j)$, for all $j \in \{1, \dots, n\}$, $i \in \{1, \dots, m\}$.

(b) X and $\tilde{X}$ admit the same copula C.

37. A bank has a loan portfolio of 100 loans. Let X_k be the default indicator for loan k such that $X_k = 1$ in case of default and 0 otherwise, for $k \in \{1, \dots, 100\}$.

(a) Supoose that X_k are independent and identically distributed with $P(X_k = 1) = 0.01$. Compute the expected value $E(N)$ of the number N of defaults and $P(N = k)$ for $k \in \{0, 1, \dots, 100\}$.

(b) Consider the risk factor Z which reflects the state of the economy. Suppose that conditional on Z the default indicators are independent and identically distributed with $P(X_k = 1|Z) = Z$, where $P(Z = 0.01) = 0.9$ and $P(Z = 0.11) = 0.1$. Compute the expected value $E(N)$ where N is defined as in (a).

(c) Consider the risk factor Z which reflects the state of the economy. Suppose that conditional on Z the default indicators are independent and identically distributed with $P(X_k = 1|Z) = Z^9$, where Z is uniformly distributed on $(0, 1)$. Compute the expected value $E(N)$ where N is defined as in (a).