Risk theory and risk management in actuarial science
Winter term 2019/2020

4th work sheet

28. Construct two random vectors $(X_1, X_2)^T$ and $(Y_1, Y_2)^T$ with different joint distributions $F_{(X_1,X_2)}$, $F_{(Y_1,Y_2)}$, respectively, such that

    (a) the variables X_1, X_2, Y_1, Y_2 are standard normally distributed, i.e. $X_1, X_2, Y_1, Y_2 \sim N(0, 1)$,

    (b) the two X-variables and the two Y-variables are uncorrelated, respectively, i.e. $\rho_L(X_1, X_2) = 0$, $\rho_L(Y_1, Y_2) = 0$, and

    (c) the α-quantiles of the corresponding sums are different, i.e. $F^{\leftarrow}_{X_1+X_2}(\alpha) \neq F^{\leftarrow}_{Y_1+Y_2}(\alpha)$ holds for some $\alpha \in (0, 1)$, where $F_{X_1+X_2}$, $F_{Y_1+Y_2}$ are the distributions of X_1+X_2 and Y_1+Y_2, respectively.

    Conclude that in general it is not possible to draw conclusions about the loss of a portfolio if only the loss distributions of the single assets in portfolio and their mutual linear correlation coefficients are known.

    Hint: Choose (X_1, X_2) to be bivariate standard normally distributed, i.e. $(X_1, X_2) \sim N_2(0, I_2)$, where 0 denotes the zero vector in $\mathbb{R}^2$ and I_2 denotes the identity matrix in $\mathbb{R}^{2\times 2}$. Choose Y_1 to be standard normally distributed, $Y_1 \sim N(0, 1)$, and set $Y_2 := VY_1$, where V is a discrete random variable independent on Y_1 with values 1 and -1 taken with probability 1/2 each.

29. (Co-monotonicity and anti-monotonicity)

    (a) Let Z be a random variable with continuous cumulative distribution function F, $Z \sim F$. Let f_1, f_2 be to strictly monotone increasing functions on $\mathbb{R}$ and let f_3 be a strictly monotone decreasing function on $\mathbb{R}$. Let $X_i = f_i(Z)$, for $i = 1, 2, 3$. Show that the Fréchet upper bound M is a copula of (X_1, X_2) and the Fréchet lower bound W is a copula of (X_1, X_3).

    (b) Let W be the (unique) copula of the random vector (X_1, X_2) with continuous marginal distributions F_1 and F_2, respectively. Show that $X_2 \overset{a.s.}{=} T(X_1)$ with $T = F_2^{\leftarrow} \circ (1 - F_1)$.

    (c) Let M be the (unique) copula of the random vector (X_1, X_2) with continuous marginal distributions F_1 and F_2, respectively. Show that $X_2 \overset{a.s.}{=} T(X_1)$ with $T = F_2^{\leftarrow} \circ F_1$.

30. (a) Prove the formula of Höffding

    $$cov(X_1, X_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (F(x_1, x_2) - F_1(x_1)F_2(x_2))dx_1 dx_2\,,$$

    for any random vector $(X_1, X_2)^T$ with c.d.f. F and marginal d.f. F_1, F_2 for which $cov(X_1, X_2) < \infty$ holds.

    Hint: Show first the identity $2cov(X_1, X_2) = E((X_1 - \bar{X}_1)(X_2 - \bar{X}_2))$, where $(\bar{X}_1, \bar{X}_2)$ is an i.i.d. copy of (X_1, X_2). Then use the identity $a - b = \int_{-\infty}^{+\infty} (\mathbb{I}_{\{b\leq x\}} - \mathbb{I}_{\{a\leq x\}})\mathrm{d}x$ and apply the latter to the pairs $X_1 - \bar{X}_1$ and $X_2 - \bar{X}_2$.

    (b) Prove the following equality for the rank correlation Spearman's rho of a random vector $(X_1, X_2)^T$ with continuous marginal distributions and unique copula C:
    $\rho_S(X_1, X_2) = 12 \int_0^1 \int_0^1 (C(u_1, u_2) - u_1u_2)du_1du_2 = 12 \int_0^1 \int_0^1 C(u_1, u_2)du_1du_2 - 3.$
    Hint: Use the equivalent definition $\rho_S(X_1, X_2) = \rho_L(F_1(X_1), F_2(X_2))$ and apply the formula of Höffding.