

Risk theory and risk management in actuarial science
Winter term 2019/2020

3d work sheet

18. Consider a portfolio consisting of the following index futures *Dow Jones Industrial Average* (\hat{DJI}), *S&P 500 ETF* (\hat{GSPC}), *Nasdaq Composite* (\hat{IXIC}), *DAX* (\hat{GDAXI}) and *ATX* (\hat{ATX}) with one piece per each index future. Estimate value at risk $\text{VaR}_{0.90}$ of the weekly losses of this portfolio in two ways as described below. Then compare and comment on the obtained results.
- (a) Use historical simulation of the losses over the last 10 years, from October 27, 2008 until October 26, 2018.
 - (b) Use the variance-covariance method, again based on the weekly logarithmic returns of the last 10 years as in (a).

The data can be downloaded from finance.yahoo.com: search for the required index future (you can well search for the abbreviations given in paranthesis above), click ‘Historical Prices’, update the ‘Time Period’ and ‘Frequency’ appropriately, and finally klick on ‘Download data’. Use the adjusted close prices to compute the weakly logarithmic returns of the single index futures.

19. Consider the daily relative returns (based on adjusted close prices) of the BMW and Siemens assets, *BMW.F* and *SIA.AS*, respectively, over the time interval October 27, 2008, and October 26, 2018. Use the Hill estimator to get an approximation of the coefficient of the regular variation and determine the corresponding estimates for $\text{VaR}_{0.90}$ and $\text{CVaR}_{0.90}$ for each of these assets. Specify a plausible range of values of k to be chosen (depending on the sample size) and generate the Hill plot for those values of k . Based on the Hill plot make a suggestion for an appropriate value of k to be used and argue your choice carefully. Use yahoo.finance.com as a data source (see Exercise 18).
20. By means of the qq-plot check whether a normal distribution or a heavy tailed distribution like the (generalized) Pareto distribution is more appropriate to model the right tail of the loss distribution of the BMW and Siemens assets as described in Exercise 19, respectively. To this end you should compare the empirical quantiles of the above mentioned losses to the (numerically or analytically) computed quantiles of the reference distributions (i.e. a normal and a generalized Pareto distribution) and summarize the results graphically as described schematically in the lecture.
21. Consider the logarithmic daily returns of the close prises of the Nasdaq Composite Index (\hat{IXIC}) and apply the method of the Hill estimator to analyse their tails. Perform the following steps with three different time intervals: (I) from November 1, 1996 till November 2, 2018, (II) from November 1, 1996 till December 26, 2008, and (III) from December 29, 2008 till November 2, 2018. Compare the obtained results and comment upon your findings.
- (a) Compare the tails of the empirical distribution of the data set to the tails of the exponential ditribution by means of the QQ-plot.
 - (b) Compute the Hill estimator for the empirical data. Argue carefully upon your choice of the threshold parameter k based on the inspection of the Hill plot as in the case of the fire insurance example discussed in the lecture.
 - (c) Based on the Hill estimator give an estimator for the $\text{VaR}_{0.95}$ and the $\text{VaR}_{0.99}$ of the data set.

The data can be downloaded from finance.yahoo.com: search for the required index (you can well search for the abbreviation given in paranthesis above), click ‘Historical Data’, update the ‘Time Period’ and ‘Frequency’ appropriately, and finally klick on ‘Download data’.

22. Use the peaks over threshold (POT) method to analyse the tails of the data described in Exercise 21.

- (a) Argue carefully upon your choice of the threshold parameter k based on the inspection of the plot of the empirical mean excess function (analogously to the case of the fire insurance example discussed in the lecture).
- (b) Maximize the log-likelihood function to obtain estimators for γ and β by using a solver of your choice. Consider the plot of the different values of the estimator $\hat{\gamma}$ of γ in dependence of the threshold parameter k to back your choice for a suitable interval of values of k (cf. the fire insurance example from the lecture).
- (c) Compute estimators for $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ for the whole interval of reasonable values of k determined in (b). Visualize the dependence of these estimators on k graphically and revise your choice for the interval of values of k , if appropriate.
- (d) Choose a value of k and visualize in one plot the empirical tail distribution and the tail distribution obtained by the POT method. Comment upon your results.

23. Let the random variables $X_i, i = 1, 2$, be such that $X_1 \sim \text{Exp}(\lambda)$ and $X_2 = t(X_1)$, where $\text{Exp}(\lambda)$ is the exponential distribution with parameter λ and $t: \mathbb{R} \rightarrow \mathbb{R}, t(x) = x^2$. Determine the coefficients of the lower and the upper tail dependence $\lambda_L(X_1, X_2), \lambda_U(X_1, X_2)$, respectively, and conclude that X_1 and X_2 have both a lower and an upper tail dependence. Compute also the coefficient of the linear correlation $\rho_L(X_1, X_2)$, compare the three computed dependence measures and comment on your results.

24. (A coherent premium principle)

Consider two constants $p > 1$ and $\alpha \in [0, 1)$. Let (Ω, \mathcal{F}, P) be some fixed probability space and \mathcal{M} be the set of all random variables L on (Ω, \mathcal{F}) for which $E(|L|^p)^{1/p}$ is finite, i.e. $E(|L|^p)^{1/p} < \infty$. Define a risk measure $\rho_{\alpha,p} := E(L) + \alpha \| (L - E(L))^+ \|_p$ on \mathcal{M} , where $\|X\|_p := E(|X|^p)^{1/p}$ is the L^p -norm of the positive part of the centered random variable $X - E(X)$ for any random variable $X \in \mathcal{M}$. Show that $\rho_{\alpha,p}$ is a coherent risk measure for any $p > 1$ and any $\alpha \in [0, 1)$. So we get a whole family of coherent risk measures $\rho_{\alpha,p}$ for $p > 1$ and $\alpha \in [0, 1)$. How do the parameters α and p influence $\rho_{\alpha,p}$? Which parameter values lead to more “conservative” risk measures?

25. (Generalized scenarios as coherent risk measures)

Denote by \mathcal{P} a set of probability measures on some underlying measurable space (Ω, \mathcal{F}) and set

$$\mathcal{M}_{\mathcal{P}} := \{L: L \text{ is a r.v. on } (\Omega, \mathcal{F}), E^Q(|L|) < \infty \text{ for all } Q \in \mathcal{P}\},$$

where $E^Q(X)$ denotes the expected value of a random variable X under the probability measure Q . Then the risk measure induced by the set of generalized scenarios \mathcal{P} is the mapping $\rho_{\mathcal{P}}: \mathcal{M}_{\mathcal{P}} \rightarrow \mathbb{R}$ such that $\rho_{\mathcal{P}}(L) := \sup\{E^Q(L): Q \in \mathcal{P}\}$. Show that $\rho_{\mathcal{P}}$ is coherent on $\mathcal{M}_{\mathcal{P}}$ for any set \mathcal{P} of probability measures on $\mathcal{M}_{\mathcal{P}}$. Interpret the scenario based risk measures (cf. lecture) as a risk measure generalized by an appropriately defined set of probability measures on appropriately defined discrete probability spaces¹.

- 26. (a) Show that $W_d(u_1, u_2, \dots, u_d) = \max\{\sum_{i=1}^d u_i - d + 1, 0\}$ is indeed a lower bound for any copula $C: [0, 1]^d \rightarrow [0, 1]$, i.e. that $W_d(u_1, u_2, \dots, u_d) \leq C(u_1, u_2, \dots, u_d)$ holds for any $d \in \mathbb{N}, d \geq 2$, any $(u_1, \dots, u_d) \in [0, 1]^d$ and any copula C as above.
- (b) Show that the Fréchet lower bound W_d is not a copula for $d \geq 3$.

Hint: Show that the rectangle inequality

$$\sum_{k_1=1}^2 \sum_{k_2=1}^2 \dots \sum_{k_d=1}^2 (-1)^{k_1+k_2+\dots+k_d} W_d(u_{1k_1}, u_{2k_2}, \dots, u_{dk_d}) \geq 0,$$

where $(a_1, a_2, \dots, a_d), (b_1, b_2, \dots, b_d) \in [0, 1]^d$ with $a_k \leq b_k$ and $u_{k1} = a_k$ and $u_{k2} = b_k$ for all $k \in \{1, 2, \dots, d\}$, is violated if $d \geq 3$ and $a_i = \frac{1}{2}, b_i = 1$, for $i = 1, 2, \dots, d$.

¹It can be shown that in the case of discrete probability spaces any coherent risk measure is induced by some set of generalized scenarios as described above, see Proposition 6.11 in A.J. McNeil, R. Frey and P. Embrechts, *Quantitative Risk Management: Concepts, Techniques and Tools*, Princeton University Press, 2005.

27. Let X_i , $i = 1, 2$, be two lognormally distributed random variables with $X_1 \sim \text{Lognormal}(0, 1)$ and $X_2 \sim \text{Lognormal}(0, \sigma^2)$, $\sigma > 0$. Compute $\rho_{L, \min}(X_1, X_2)$ and $\rho_{L, \max}(X_1, X_2)$ in dependence of σ and compare their values for different values of $\sigma > 0$. What can you say about the copula of (X_1, X_2) in each of the cases? Plot the graphs of $\rho_{L, \min}(X_1, X_2)$ and $\rho_{L, \max}(X_1, X_2)$ as functions of σ and comment on the behaviour of these functions for $\sigma \rightarrow +\infty$?

Hint: Consider $X_1 = \exp(Z)$ and $X_2 = \exp(\sigma Z)$ or $X_2 = \exp(-\sigma Z)$ for a standard normally distributed random variable Z .