Risk theory and management in actuarial science
winter term 2019/20
2nd work sheet

8. Investigate whether the following functions are slowly varying:

   (a) $L(x) := \ln(1 + \ln(1 + x))$

   (b) $f(x) := 3 + \sin x$

   (c) $f(x) := \ln(e + x) + \sin x$

9. Show that the following distributions are regularly varying:

   (a) The Pareto distribution G_α with parameter α given as $G_\alpha(x) = 1 - x^{-\alpha}$, for $x > 1$, where $\alpha > 0$. Show that $\bar{G}_\alpha(tx)/\bar{G}_\alpha(t) = x^{-\alpha}$ holds for $t > 0$, $x > 0$, thus $\bar{G}_\alpha \in RV_{-\alpha}$.

   (b) The Fréchet distribution Φ_α with parameter α given as $\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}$ for $x > 0$ and $\Phi_\alpha(0) = 0$, where $\alpha > 0$. Show that $\lim_{x\to\infty} \bar{\Phi}_\alpha(x)/x^{-\alpha} = 1$, i.e. $\bar{\Phi}_\alpha \in RV_{-\alpha}$.

10. Let X and Y be positive random variables representing losses in two lines of business (e.g. losses due to fire and car accidents) of an insurance company. Suppose that X has distribution function F which satisfies $\bar{F} \in RV_{-\alpha}$ for $\alpha > 0$. Moreover suppose that Y has finite moments of all orders, i.e. $E(Y^k) < \infty$, for every $k > 0$. Compute $\lim_{x\to\infty} P(X > x|X + Y > x)$, i.e. the asymptotic probability of a large loss in the fire insurance line given a large total loss.

11. Prove the following characterization of the maximum domain of attraction of an extreme value distribution H (also fomulated in the lecture):
    $F \in MDA(H)$ with normalizing and centralizing constants $a_n > 0$, b_n, $n \in \mathbb{N}$, respectively, iff $\lim_{n\to\infty} n\bar{F}_n(a_n x + b) = -\ln(H(x))$, for all $x \in \mathbb{R}$.

12. (The Maxima of the Poisson distribution)
    Let $X \sim P(\lambda)$, i.e. $P(X = k) = e^{-\lambda}\lambda^k/k!$, $k \in \mathbb{N}_0$, for some parameter $\lambda > 0$. Show that there exists no extreme value distribution Z such that $X \in MDA(Z)$.

    Hint: Use the following lemma of Leadbetter et al.[1].
    For any discrete non-negative distribution F with right end $x_F = +\infty$ (i.e. a random variable with distribution F can take arbitrarily large values), the following two statements are equivalent for every $\tau \in (0, \infty)$: a) there exists a sequence $u_n \in \mathbb{R}$, $n \in \mathbb{N}$ such that $\lim_{n\to\infty} n\bar{F}(u_n) = \tau$, and b) $\lim_{n\to\infty} \frac{\bar{F}(n)}{\bar{F}(n-1)} = 1$.

    You don't need to prove the lemma.

13. (Maximum domain of attraction of the Fréchet distribution)
    Show that the following distributions belong to the maximum domain of attraction $MDA(\Phi_\alpha)$ of the Fréchet distribution Φ_α, for some $\alpha > 0$, and determine the normalizing and centralizing constants $a_n > 0$, b_n, for $n \in \mathbb{N}$, respectively.

    (a) The Pareto distribution with parameter $\alpha > 0$: $G_\alpha(x) = 1 - x^{-\alpha}$, for $x > 1$.

    (b) The Cauchy distribution with density function $f(x) = (\pi(1 + x^2))^{-1}$, $x \in \mathbb{R}$.

    (c) The Students distribution with parameter $\alpha \in \mathbb{N}$ and density function $f(x) = \frac{\Gamma((\alpha+1)/2)}{\sqrt{\alpha\pi}\Gamma(\alpha/2)(1+x^2/\alpha)^{(\alpha+1)/2}}$, $\alpha \in \mathbb{N}$, $x \in \mathbb{R}$.

[1] M.R. Leadbetter, G. Lindgren and H. Rootzen, *Extremes and related properties of random sequences and processes*, Springer, Berlin, 1983.

14. (Maximum domain of attraction of the Weibull ditribution)
    Let $X \sim U(0,1)$ be uniformly distributed on $[0,1]$. Show that X belongs to the maximum domain of attraction of the Weibull distribution with parameter 1, i.e. $X \in MDA(\Psi_1)$, with normalizing constant $a_n = 1/n$, for $n \in \mathbb{N}$.

15. (Maximum domain of attraction of the Gumbel ditribution)
    Check whether the following distributions belong to the maximum domain of attraction $MDA(\Lambda)$ of the Gumbel distribution.

    (a) The normal distribution $F(x) = (2\pi)^{-1/2} \exp\{-x^2/2\}$, $x \in \mathbb{R}$.

    (b) The exponential distribution with density function $f(x) = \lambda^{-1} \exp\{-\lambda x\}$, $x > 0$, for some parameter $\lambda > 0$.

    (c) The lognormal distribution with density function $f(x) = (2\pi x^2)^{-1/2} \exp\{-(\ln x)^2/2\}$, $x > 0$.

16. (a) Let X be a random variable with distribution function F. Derive an equation relating $VaR_\alpha(X)$, $CVaR_\alpha(X)$ and $e_X(q_\alpha)$, where $q_\alpha := VaR_\alpha(X)$ and $\alpha \in (0,1)$.

    (b) Compute the mean excess function of the exponential distribution, i.e. compute $e_X(u)$ for $X \sim Exp(\lambda)$ and $u \geq 0$.

    (c) Use the result form (a) to compute $CVaR_\alpha(X)$ in the case of the exponential distribution, i.e. for $X \sim Exp(\lambda)$, $\alpha \in (0,1)$.

17. (a) Compute the expectation of $E(G_{\gamma,0,\beta})$ of the generalized Pareto distribution $G_{\gamma,0,\beta}$.

    (b) Consider a random variable X with distribution function X for which the approximation $\bar{F}_u(x) \approx \bar{G}_{\gamma,0,\beta(u)}(x)$ holds with $\gamma \notin \{0,1\}$. Show that this implies the approximation $\bar{F}_v(x) \approx \bar{G}_{\gamma,0,\beta(u)+\gamma(v-u)}(x)$, $\forall v \geq u$.

    (c) Use the results of (a) and (b) to show that $e_X(v)$ is linear in v for $v \geq u$ for any fixed threshold $u > 0$.