Risk theory and management in actuarial science
winter term 2019/20

1st work sheet

1. Consider two random losses L_1, L_2 distributed as follows: L_1 is normally distributed as $L_1 \sim N(0, 2)$ and L_2 has a t-distribution with $m = 4$ degrees of freedom, i.e. $L_2 \sim t_4$. Observe that the variances of L_1 and L_2 coincide, i.e. $\sigma^2(L_1) = 2$ and $\sigma^2(L_2) = \frac{m}{m-2} = 2$ hold. By computing the loss probabilities $\mathbb{P}(\mathrm{L_2 > x})$, $\mathbb{P}(\mathrm{L_1 > x})$ and by plotting the logarithm of the quotient $\ln[\mathbb{P}(\mathrm{L_2 > x})/\mathbb{P}(\mathrm{L_1 > x})]$ show that the loss probability in the case of L_2 is much larger than the loss probability in the case of L_1.

2. Let $L \sim N(\mu, \sigma^2)$. Show that $VaR_\alpha(L) = \mu + \sigma q_\alpha(\Phi) = \mu + \sigma\Phi^{-1}(\alpha)$ holds, where Φ is the distribution function of a random variable $X \sim N(0, 1)$. Further show that $CVaR_\alpha(X) = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ holds, where ϕ is the density function of X as above, and derive also a formula for $CVaR_\alpha(L)$.

3. Consider a portfolio consisting of 5 pieces of an asset A. The today's price of A is $S_0 = 100$. The daily logarithmic returns are i.i.d.: $X_1 = \ln\frac{S_1}{S_0}$, $X_2 = \ln\frac{S_2}{S_1}, \ldots \sim N(0, 0.01)$. Let L_1 be the 1-day portfolio loss in the time interval (today, tomorrow).

   (a) Compute $VaR_{0.99}(L_1)$.

   (b) Compute $VaR_{0.99}(L_{100})$ and $VaR_{0.99}(L^{\Delta}_{100})$, where L_{100} is the 100-day portfolio loss over a horizon of 100 days starting with today. L^{Δ}_{100} is the linearization of the above mentioned 100-day PF-portfolio loss. Compare the two values and comment on the results.

   Hint: Use the equality $\Phi^{-1}(0.99) \approx 2.3$, where Φ is the distribution function of a random variable $X \sim N(0, 1)$.

4. (a) Let $L \sim Exp(\lambda)$. Compute $CVaR_\alpha(L)$.

   (b) Let the distribution function F_L of the loss function L be given by $F_L(x) = 1 - (1 + \gamma x)^{-1/\gamma}$ for $x \geq 0$ and some parameter $\gamma \in (0, 1)$ (this is the generalized Pareto distribution). Compute $CVaR_\alpha(L)$.

5. Let the loss L be distributed according to the Students t-distribution with $\nu > 1$ degrees of freedom. The density function of L is given as

$$g_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)}\left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}$$

   Show that $CVaR_\alpha(L) = \frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha}\left(\frac{\nu+(t_\nu^{-1}(a))^2}{\nu-1}\right)$, where t_ν is the distribution function of L.

6. Consider an i.i.d. sample $x_1, x_2, \ldots, x_n$ sorted decreasingly, i.e. $x_n < x_{n-1} \ldots < x_1$, from a common unknown continuous distribution function F. Assume that we want to compute a confidence interval for $q_\alpha(F)$ with confidence level $p' \geq p$. Use the alternative method without bootstrapping (cf. lecture) to determine the indices $l_{\alpha,p}, u_{\alpha,p} \in \{1, 2, \ldots, n\}$ of the data points from the sample which yield the smallest possible confidence intervals $(x_{l_{\alpha,p}}, x_{u_{\alpha,p}})$ for $\alpha \in \{0.8, 0.9\}$ and $p \in \{0.80, 0.90\}$. Repeat your computations for $n = 50$, $n = 100$ and $n = 200$. What kind of mutual dependencies can you observe between the computed confidence intervals and the sample size n?

7. Consider a portfolio consisting of one piece of the 10 Year Treasury Bond with holding time 1 month. Let L be its loss function. Assume that the portfolio is evaluated on the first of every month. Use the historical yield values from November 1, 2004 to October 1, 2019[1] to compute the historically realized losses of this portfolio over the time period of 15 years mentioned above. Compute the empirical estimators of $VaR_{0.9}(L)$, $CVaR_{0.9}(L)$. Further use the bootstrapping method to compute a confidence interval with confidence level $p = 0.8$ for each of the two quantities above.

[1] These historical data are available at yahoo finance, more precisely at `https://finance.yahoo.com/quote/%5ETNX/history?period1=1254607200&period2=1570140000&interval=1mo&filter=history&frequency=1mo`