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Let $X = (X_1, X_2, \ldots, X_d) = \mu + AY$ be elliptically distributed with $\mu \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times k}$ and a spherically distributed vector $Y \sim S_k(\psi)$. Assume that $0 < E(X_k^2) < \infty$ holds $\forall k$. If the risk measure ρ has the properties (C1) and (C3) and $\rho(Y_1) > 0$ for the first component Y_1 of Y, then

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Theorem: (Embrechts et al., 2002)
Let M be the set of returns of the portfolii in $\mathcal{P} := \{w = (w_i) \in \mathbb{R}^d, \sum_{i=1}^d |w_i| = 1\}$. Let the asset returns $X = (X_1, X_2, \ldots, X_d)$ be elliptically distributed, $X = (X_1, X_2, \ldots, X_d) \sim E_d(\mu, \Sigma, \psi)$ for some $\mu \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d\times d}$ and $\psi\colon \mathbb{R} \to \mathbb{R}$. Then VaR_α ist coherent in M, for any $\alpha \in (0.5, 1)$.

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Equivalently, a copula C is a function $C\colon [0,1]^d \to [0,1]$, with the following properties:

1. $C(u_1, u_2, \ldots, u_d)$ is mon. increasing in each variable u_i, $1 \le i \le d$.
2. $C(1, 1, \ldots, 1, u_k, 1, \ldots, 1) = u_k$ for any $k \in \{1, \ldots, d\}$ and $\forall u_k \in [0,1]$.
3. The *rectangle inequality* holds $\forall (a_1, a_2, \ldots, a_d) \in [0,1]^d$, $\forall (b_1, b_2, \ldots, b_d) \in [0,1]^d$ with $a_k \le b_k$, $\forall k \in \{1, 2, \ldots, d\}$:

$$\sum_{k_1=1}^{2} \ldots \sum_{k_d=1}^{2} (-1)^{k_1+k_2+\ldots+k_d} C(u_{1k_1}, u_{2k_2}, \ldots, u_{dk_d}) \ge 0\,,$$

where $u_{j1} = a_j$ and $u_{j2} = b_j$.

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Remark: The k-dimensional marginal distributions of a d-dimensional copula are k-dimensional copulas, for all $2 \le k \le d$.

Lemma: Let $h\colon \mathbb{R} \to \mathbb{R}$ be a monotone increasing function with $h(\mathbb{R}) = \mathbb{R}$ and $h^{\leftarrow}\colon \mathbb{R} \to \mathbb{R}$ be the generalized inverse function of h. Then the following statements hold:

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Lemma: Let X be a r.v. with continuous distribution function F. Then $P\left(F^{\leftarrow}(F(x)) = x\right) = 1$, i.e. $F^{\leftarrow}(F(X)) \overset{a.s.}{=} X$

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Theorem: Let G be a d.f. in $\mathbb{R}$. The following statements holds

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Let $F\colon \mathbb{R}^d \to [0,1]$ a c.d.f. with marginal d.f. $F_1,\ldots,F_d$. There exists a copula C, such that for all $x_1, x_2, \ldots, x_d \in \bar{\mathbb{R}} = [-\infty, \infty]$ the equality

$$F(x_1, x_2, \ldots, x_d) = C(F_1(x_1), F_2(x_2), \ldots, F_d(x_d)) \text{ holds.}$$

If $F_1,\ldots,F_d$ are continuous, then C is unique.

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Vice-versa, if C is a copula and F_1,...,F_d are d.f., then the function F defined by the equality above is a c.d.f. with marginal d.f. F_1,...,F_d.

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C as above is called *the copula of* F. For a random vector $X \in \mathbb{R}^d$ with c.d.f. F we say that C is *the copula of* X.

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Corollary: Let F be a c.d.f. with continuous marginal d.f. $F_1, \ldots, F_d$. The unique copula C of F is given as :

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Theorem: (Copula invariance w.r.t. strictly monotone transformations) Let $X = (X_1, X_2, \ldots, X_d)^T$ be a random vector with continuous marginal d.f. $F_1, F_2, \ldots, F_d$ and copula C. Let T_1,T_2,..., T_d be strictly monotone increasing functions in $\mathbb{R}$. Then C is also the copula of $(T_1(X_1), T_2(X_2), \ldots, T_d(X_d))^T$.

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Example: Let $X = (X_1, \ldots, X_d) \sim N_d(0, \Sigma)$ with $\Sigma = R$ being the correlation matrix of X. Let ϕ_R and ϕ be the c.d.f of X and X_1, resp.. The copula of X is called *a Gaussian copula* and is denoted by C_R^{Ga}:

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C_R^{Ga} is the copula of any non-degenerate normal distribution $N_d(\mu, \Sigma)$ with correlation matrix R.

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For $d = 2$ and $\rho = R_{12} \in (-1, 1)$ we have :

$$C_R^{Ga}(u_1, u_2) = \int_{-\infty}^{\phi^{-1}(u_1)} \int_{-\infty}^{\phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{ \frac{-(x_1^2 - 2\rho x_1 x_2 + x_2^2)}{2(1-\rho^2)} \right\} dx_1 dx_2$$