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1. $X \in \mathbb{R}^d$ has a spherical distribution.
2. There exists a function $\psi \colon \mathbb{R} \to \mathbb{R}$ of a scalar variable, such that the characteristic function of X satisfies

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3. For every vector $a \in \mathbb{R}^d$, $a^t X \stackrel{d}{=} ||a|| X_1$ holds, where $||a||^2 = a_1^2 + a_2^2 + \ldots + a_d^2$.

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4. X has the stochastic representation $X \stackrel{d}{=} RS$, where $S \in \mathbb{R}^d$ is a random vector uniformly distributed on the unit sphere S^{d-1}, $\mathcal{S}^{d-1} := \{x \in \mathbb{R}^d \colon ||x|| = 1\}$, and $R \geq 0$ is a r.v. independent of S.

Notation: $X \sim S_d(\psi)$, cf. 2.

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Example: The standard normal distribution is a spherical distribution.

Let $X \sim N_d(0, I)$. Then $X \sim S_d(\psi)$ mit $\psi = \exp(-x/2)$.
Indeed, $\phi_X(t) = \exp\{it^T 0 - \frac{1}{2} t^T I t\} = \exp\{-t^T t/2\} = \psi(t^T t)$, and thus X has a spherical distribution.

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(i) Simulate s from S which is uniformly distributed on the unit sphere $\mathcal{S}^{d-1}$ (e.g. by simulating y from a multivariate standard normal distribution $Y \sim N_d(0, I)$ and then setting $s = y/||y||$).

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The characteristic function can be written as

$$\begin{aligned}\phi_X(t) = E(\exp\{it^TX\}) = E(\exp\{it^T(\mu + AY)\}) \\ = \exp\{it^T\mu\}E(\exp\{i(A^Tt)^TY\}) \\ = \exp\{it^T\mu\}\psi(t^T\Sigma t),\end{aligned}$$

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IF $A \in \mathbb{R}^{d \times d}$ is nonsingular, then we have the following relation between elliptical and spherical distributions:
$X \sim E_d(\mu, \Sigma, \psi) \Leftrightarrow A^{-1}(X - \mu) \sim S_d(\psi)$, $A \in \mathbb{R}^{d \times d}$, $AA^T = \Sigma$.

Elliptical distributions (contd.)

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Theorem: (Stochastic representation of elliptical distributions)
Let $X \in \mathbb{R}^d$ be an d-dimensional random vector. $X \sim E_d(\mu, \Sigma, \psi)$ iff $X \stackrel{d}{=} \mu + RAS$, where $S \in \mathbb{R}^k$ is a random vector uniformly distributed on the unit sphere $\mathcal{S}^{k-1}$, $R \geq 0$ is a r.v. independent of S, $A \in \mathbb{R}^{d \times k}$ is a constant matrix with $\Sigma = AA^T$ and $\mu \in \mathbb{R}^d$ is a constant vector.

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Examples of elliptical distributions

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- Multivariate normal distribution

  Let $X \sim N(\mu, \Sigma)$ with Σ positive definite. Then for $A \in \mathbb{R}^{d \times k}$ with $AA^T = \Sigma$ we have $X \stackrel{d}{=} \mu + AZ$, where $Z \in N_k(0, I)$. Moreover $Z = RS$ holds with S being uniformly distributed on the unit sphere $\mathcal{S}^{k-1}$ and $R^2 \sim \chi^2_k$. Thus $X \stackrel{d}{=} \mu + RAS$ holds and hence $X \sim E_d(\mu, \Sigma, \psi)$ with $\psi(x) = \exp\{-x/2\}$.

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- Multivariate normal variance mixtures

  Let $Z \sim N_d(0, I)$. Then Z has a spherical distribution with stochastic representation $Z \stackrel{d}{=} VS$ with $V^2 = ||Z||^2 \sim \chi_d^2$. Let $X = \mu + WAZ$ be a variance normal mixture distribution. Then we get $X \stackrel{d}{=} \mu + VWAS$ with $V^2 \sim \chi_d^2$ and VW is a nonnegative r.v. independent of S. Thus X is elliptically distributed with $R = VW$.

Properties of elliptical distributions

Theorem:

Let $X \sim E_k(\mu, \Sigma, \psi)$. X has the following properties:

- Linear transformation

  For $B \in \mathbb{R}^{k \times d}$ and $b \in \mathbb{R}^k$ we have:

$$BX + b \in E_k(B\mu + b, B\Sigma B^T, \psi).$$

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- Marginal distributions

  Set $X^T = \left({X^{(1)}}^T, {X^{(2)}}^T \right)$ for ${X^{(1)}}^T = (X_1, X_2, \ldots, X_n)^T$ and ${X^{(2)}}^T = (X_{n+1}, X_{n+2}, \ldots, X_k)^T$ and analogously set $\mu^T = \left({\mu^{(1)}}^T, {\mu^{(2)}}^T \right)$ as well as $\Sigma = \begin{pmatrix} \Sigma^{(1,1)} & \Sigma^{(1,2)} \\ \Sigma^{(2,1)} & \Sigma^{(2,2)} \end{pmatrix}$. Then $X_1 \sim E_n\left(\mu^{(1)}, \Sigma^{(1,1)}, \psi\right)$ and $X_2 \sim E_{k-n}\left(\mu^{(2)}, \Sigma^{(2,2)}, \psi\right)$.

Properties of elliptical distributions (contd.)

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- Conditional distributions

  Assume that Σ is nonsingular. Then

  $X^{(2)} \Big| X^{(1)} = x^{(1)} \sim E_{d-k}\left(\mu^{(2,1)}, \Sigma^{(22,1)}, \tilde{\psi}\right)$ where

  $\mu^{(2,1)} = \mu^{(2)} + \Sigma^{(2,1)}\left(\Sigma^{(1,1)}\right)^{-1}\left(x^{(1)} - \mu^{(1)}\right)$ and

  $\Sigma^{(22,1)} = \Sigma^{(2,2)} - \Sigma^{(2,1)}\left(\Sigma^{(1,1)}\right)^{-1}\Sigma^{(1,2)}.$

  Typically $\tilde{\psi}$ is a different characteristic generator than the original ψ (see Fang, Katz and Ng 1987).

Properties of elliptical distributions (contd.)

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- Quadratic forms

  If Σ is nonsingular, then $D^2 = (X-\mu)^T \Sigma^{-1}(X-\mu) \sim R^2$, where R is the nonnegative r.v. in the stochastic representation $Y = RS$ of the spherical distribution Y with $S \sim U\left(\mathcal{S}^{(d-1)}\right)$ and $X = \mu + AY$.

  The random variable D is called *Mahalanobis distance*.

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- Convolutions

  Let $X \sim E_k(\mu, \Sigma, \psi)$ and $Y \sim E_k(\tilde{\mu}, \Sigma, \tilde{\psi})$ be two independent random vectors. Then $X + Y \sim E_k(\mu + \tilde{\mu}, \Sigma, \bar{\psi})$ where $\bar{\psi} = \psi\tilde{\psi}$.

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Important: $X \sim E_k(\mu, I_k, \psi)$ does not imply that the components of X are independent. The components of X are independent iff X is multivariate normally distributed with the unit matrix as a covariance matrix.

Coherent risk measures

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(C4) Monotonicity:

$\forall X_1, X_2 \in M$ the implication $X_1 \overset{a.s.}{\leq} X_2 \Longrightarrow \rho(X_1) \leq \rho(X_2)$ holds.

Convex risk measures

Consider the property:

(C5) Convexity:

$\forall X_1, X_2 \in M$, $\forall \lambda \in [0,1]$

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Definition: A risk measure ρ in M with the properties (C1),(C4) and (C5) is called *convex* in M.

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Definition: A risk measure ρ in M with the properties (C1),(C4) and (C5) is called *convex* in M.

Observation: VaR is not coherent in general.

Convex risk measures

Consider the property:

(C5) Convexity:
$\forall X_1, X_2 \in M$, $\forall \lambda \in [0,1]$
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Observation: VaR is not coherent in general.
Let the probability measure P be defined by some continuous or discrete probabilty distribution F.
$VaR_\alpha(F) = F^{\leftarrow}(\alpha)$ has the properties (C1), (C3) and (C4), whereas the subadditivity (C2) is not fulfilled in general.

Coherent risk measures (contd.)

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Example: Let the probability measure P be defined by the binomial distribution $B(p, n)$ for $n \in \mathbb{N}$, $p \in (0, 1)$. We show that $VaR_\alpha(B(p, n))$ is not subadditive.

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Theorem: Let $(\Omega, \mathcal{F}, P)$ be a probability space and $M \subseteq L^{(0)}(\Omega, \mathcal{F}, P)$ be the set of the random variables with a continuous distribution in $(\Omega, \mathcal{F}, P)$. $CVaR_\alpha$ is a coherent risk measure in M, $\forall \alpha \in (0, 1)$.

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To show (C2) observe that for a sequence of i.i.d. r.v. L_1, L_2, ..., L_n with order statistics $L_{1,n} \geq L_{2,n} \geq \ldots \geq L_{n,n}$ and for any $m \in \{1, 2, \ldots, n\}$

$$\sum_{i=1}^{m} L_{i,n} = \sup\{L_{i_1} + L_{i_2} + \ldots + L_{i_m} : 1 \leq i_1 < \ldots < i_m \leq n\} \text{ holds.}$$

The mean-risk portfolio optimization model

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Let $\mathcal{P}_m$ be the family of portfolios in $\mathcal{P}$ with $E(Z(w)) = m$, for some $m \in \mathbb{R}$, $m > 0$.
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For a risk measure ρ the *mean-ρ portfolio optimization model* is:

$$\min_{w \in \mathcal{P}_m} \rho(Z(w)) \tag{1}$$

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\min_{w} \quad & w^T \Sigma w \\
\text{s.t.} \quad & \\
& w^T \mu = m \\
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\end{aligned}
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If $\rho = VaR_\alpha$, $\alpha \in (0,1)$ we get the *mean-VaR pf. optimization model*

$$\min_{w \in \mathcal{P}_m} VaR_\alpha(Z(w)).$$

Question: What is the relationship between these three portfolio optimization models?