Characterization of the domain of attraction

(i) Let ϕ be the standard normal distribution function. The equivalence

$$F \in DA(\phi) \Longleftrightarrow \lim_{x \to \infty} \frac{x^2 \int_{[-x,x]^C} dF(y)}{\int_{[-x,x]} y^2 dF(y)} = 0$$

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Remark: Let $F \in DA(G_\alpha)$ for $\alpha \in (0,2)$. Then $E(|X|^\delta) < \infty$ for $\delta < \alpha$ and $E(|X|^\delta) = \infty$ for $\delta > \alpha$ hold.

Limit distributions of normalized and centered maxima

Let (X_k), $k \in \mathbb{N}$, be non-gegenerate i.i.d. r.v. with distribution function F.

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Consider $\lim_{n\to\infty} P(a_n^{-1}(M_n - b_n) \leq x) = \lim_{n\to\infty} P(M_n \leq u_n)$, where $u_n = a_n x + b_n$, $\forall n \in \mathbb{N}$.

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Let $\tau \in [0, \infty]$ and a sequence of reals $u_n \in \mathbb{R}$. Then the following holds

$$\lim_{n\to\infty} n\bar{F}(u_n) = \tau \iff \lim_{n\to\infty} P(M_n \leq u_n) = exp\{-\tau\}.$$

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Remark: The convergence to types theorem implies that H and $\tilde{H}$ are of the same type, if
$\lim_{n\to\infty} a_n^{-1}(M_n - b_n) = H$ and $\lim_{n\to\infty} \tilde{a}_n^{-1}(M_n - \tilde{b}_n) = \tilde{H}$.

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Theorem: (Fischer-Tippet Theorem, Proof in Resnick 1987, page 9-11)
Let (X_k) be a sequence of i.i.d. r.v.. If the constants $a_n, b_n \in \mathbb{R}$, $a_n > 0$, and a non-degenerate disribution H exist, such that $\lim_{n\to\infty} a_n^{-1}(M_n - b_n) = H$, then H is of the same type as one of the following three distributions:

$$\begin{array}{lll}
\text{Fréchet} & \Phi_\alpha(x) = \begin{cases} 0 & x \leq 0 \\ \exp\{-x^{-\alpha}\} & x > 0 \end{cases} & \alpha > 0 \\
\text{Weibull} & \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & x \leq 0 \\ 1 & x > 0 \end{cases} & \alpha > 0 \\
\text{Gumbel} & \Lambda(x) = \exp\{-e^{-x}\} & x \in \mathbb{R}
\end{array}$$

Extreme value distributions

The distributions Φ_α, Ψ_α and Λ are called *standard extreme value distributions (standard evd)*. The distributions which are of the same type as the standard evd are called *extreme value distributions* (evd).

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Definition: We say that the r.v. X (or the corresponding distribution) belongs to the *maximum domain of attraction* of the evd H iff there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n^{-1}(M_n - b_n) = H$ holds. Notation: $X \in MDA(H)$ ($F \in MDA(H)$).

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Theorem: (Characterisation of MDA, proof is left as an exercise) $F \in MDA(H)$ with normalizing and centering constants $a_n > 0$ snd $b_n \in \mathbb{R}$ holds, iff

$$\lim_{n\to\infty} n\bar{F}(a_n x + b_n) = -\ln H(x), \forall x \in \mathbb{R},$$

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There exist distributions which do not belong to the MDA of any evd!
Example: (The Poisson distribution)
Let $X \sim P(\lambda)$, i.e. $P(X = k) = e^{-\lambda}\lambda^k/k!$, $k \in \mathbb{N}_0$, $\lambda > 0$. Show that there exist no evd Z such that $X \in MDA(Z)$.

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Definition: (The generalized extreme value distribution (gevd))
Let the distribution function H_γ be given as follows:

$$H_\gamma(x) = \begin{cases} \exp\{-(1+\gamma x)^{-1/\gamma}\} & \text{wenn } \gamma \neq 0 \\ \exp\{-\exp\{-x\}\} & \text{wenn } \gamma = 0 \end{cases}$$

where $1 + \gamma x > 0$, i.e. the definition area of H_γ is given as

$$\begin{array}{ll} x > -\gamma^{-1} & \text{wenn } \gamma > 0 \\ x < -\gamma^{-1} & \text{wenn } \gamma < 0 \\ x \in \mathbb{R} & \text{wenn } \gamma = 0 \end{array}$$

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Theorem: (Characterisation of $MDA(H_\gamma)$)

- $F \in MDA(H_\gamma)$ with $\gamma > 0 \iff F \in MDA(\Phi_\alpha)$ with $\alpha = 1/\gamma > 0$.
- $F \in MDA(H_0) \iff F \in MDA(\Lambda)$.
- $F \in MDA(H_\gamma)$ with $\gamma < 0 \iff F \in MDA(\Psi_\alpha)$ with $\alpha = -1/\gamma > 0$.

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Observation: $\lim_{x\to+\infty} \frac{\bar{\Phi}_\alpha(x)}{x^{-\alpha}} = 1$, $\forall \alpha > 0$. Thus for $\Phi_\alpha \in MDA(\Phi_\alpha)$ we have $\bar{\Phi}_\alpha \in RV_{-\alpha}$. Does this generally hold for members of $MDA(\Phi_\alpha)$?

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Examples: The following distributions belong to $MDA(\Phi_\alpha)$:

- Pareto: $F(x) = 1 - x^{-\alpha}$, $x > 1$, $\alpha > 0$.
- Cauchy: $f(x) = (\pi(1 + x^2))^{-1}$, $x \in \mathbb{R}$.
- Student: $f(x) = \frac{\Gamma((\alpha+1)/2)}{\sqrt{\alpha\pi}\Gamma(\alpha/2)(1+x^2/\alpha)^{(\alpha+1)/2}}$, $\alpha \in \mathbb{N}$, $x \in \mathbb{R}$.
- Loggamma: $f(x) = \frac{\alpha^\beta}{\Gamma(\beta)}(\ln x)^{\beta-1}x^{-\alpha-1}$, $x > 1$, $\alpha, \beta > 0$.

Characterisations of MDAs (contd.)

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Example: Let $X \sim U(0,1)$. it holds $X \in MDA(\Psi_1)$ with $a_n = 1/n$, $n \in \mathbb{N}$.