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- analytical solution
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Disadvantages:

- Linearisation is not always appropriate, only for a short time horizon reasonable
- The normal distribution assumption could lead to underestimation of risks and should be argued upon (e.g. in terms of historical data)

(iii) Monte-Carlo approach

(1) historical observations of risk factor changes $X_{m-n+1}, \ldots, X_m$.

(2) assumption on a parametric model for the cumulative distribution function of X_k, $m-n+1 \leq k \leq m$;
e.g. a common distribution function F and independence

(3) estimation of the parameters of F.

(4) generation of N samples $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N$ from F ($N \gg 1$) and computation of the losses $l_k = l_{[m]}(\tilde{x}_k)$, $1 \leq k \leq N$

(5) computation of the empirical distribution of the loss function L_{m+1}:

$$\hat{F}_N^{L_{m+1}}(x) = \frac{1}{N} \sum_{k=1}^{N} I_{[l_k,\infty)}(x).$$

(5) computation of estimates for the VaR and CVAR of the loss function: $\widehat{VaR}(L_{m+1}) = \left(\hat{F}_N^{L_{m+1}}\right) = l_{[N(1-\alpha)]+1,N}$,

$\widehat{CVaR}(L_{m+1}) = \frac{\sum_{k=1}^{[N(1-\alpha)]+1} l_{k,N}}{[N(1-\alpha)]+1}$,

where the losses are sorted $l_{1,N} \geq l_{2,N} \geq \ldots \geq l_{N.N}$.

Monte-Carlo approach (contd.)

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- very flexible; can use any distribution F from which simulation is possible
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Disadvantages:

- computationally expensive; a large number of simulations needed to obtain good estimates

Monte-Carlo approach (contd.)

Example

The portfolio consists of one unit of asset S with price be S_t at time t. The risk factor changes

$$X_{k+1} = \ln(S_{t_{k+1}}) - \ln(S_{t_k}),$$

are i.i.d. with distribution function F_θ for some unknown parameter θ.

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Depending on F_θ it can be complicated or impossible to compute CVaR analytically.
Alternative: Monte-Carlo simulation.

Monte-Carlo approach (contd.)

Example

Let the portfolio and the risk factor changes X_{k+1} be as in the previous example.
A popular model for the logarithmic returns of assets is GARCH(1,1) (see e.g. Alexander 2002):

$$X_{k+1} = \sigma_{k+1} Z_{k+1} \quad (1)$$

$$\sigma_{k+1}^2 = a_0 + a_1 X_k^2 + b_1 \sigma_k^2 \quad (2)$$

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It is simple to simulate from this model.

However analytical computation of VaR and CVaR over a certain time interval consisting of many periods is cumbersome! Check it out!

Chapter 3: Extreme value theory

Notation:

- We will often use the same notation for the distribution of a random variable (r.v.) and its (cumulative) distribution function!
- $f(x) \sim g(x)$ for $x \to \infty$ means $\lim_{x\to\infty} f(x)/g(x) = 1$
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These two “definitions” are not equivalent!

Regular variation

Definition

A measurable function $h\colon (0,+\infty) \to (0,+\infty)$ *has a regular variation with index* $\rho \in \mathbb{R}$ *towards* $+\infty$ *iff*

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If $\rho < 0$, *then the convergence in (3) is uniform in every interval* $(b,+\infty)$ *for* $b > 0$.

Example

Show that $L \in RV_0$ *holds for the functions L as below:*

(a) $\lim_{x\to+\infty} L(x) = c \in (0,+\infty)$

(b) $L(x) := \ln(1+x)$

(c) $L(x) := \ln(1+\ln(1+x))$

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Notice: a function $L \in RV_0$ can have an infinite variation on ∞, i.e.

$$\lim \inf_{x \to \infty} L(x) = 0 \text{ and } \lim \sup_{x \to \infty} L(x) = \infty\,,$$

as for example $L(x) = \exp\{(\ln(1 + x))^2 \cos((\ln(1 + x))^{1/2})\}$.

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Proposition (no proof)
Let $X > 0$ be a r.v. with distribution function F, such that $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$. Then $E(X^\beta) < \infty$ for $\beta < \alpha$ and $E(X^\beta) = \infty$ for $\beta > \alpha$ hold.

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The converse is not true!

Application of regular variation

Example 1: Let X_1 and X_2 be two continuous nonnegative i.i.d. r.v. with distribution function F, $\bar{F} \in RV_{-\alpha}$ for some $\alpha > 0$. Let X_1 (X_2) represent the loss of a portfolio which consists of 1 unit of asset A_1 (A_2).

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Consider a portfolio P_1 containing 2 units of asset A_1 and a portfolio P_2 containing one unit of A_1 and one unit of A_2. Let L_i represent the loss of portfolio P_i, $i = 1, 2$.

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Compare the probabilities of high losses in the two portfolios by computing the limit

$$\lim_{l \to \infty} \frac{Prob(L_2 > l)}{Prob(L_1 > l)}.$$

In which cases are the extreme losses of the diversified portfolio smaller then those of the non-diversified portfolio?

Application of regular variation (contd.)

Example 2: Let $X, Y \geq 0$ be two r.v. which represent the losses of two business lines of an insurance company (e.g. fire and car accidents).

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Compute $\lim_{x \to \infty} P(X > x | X + Y > x)$.

Classical extreme value theory

Let (X_k), $k \in \mathbb{N}$, be non-degenerate i.i.d. r.v. with distribution function F. For $n \geq 1$ define $S_n := \sum_{i=1}^{n} X_i$ and $M_n := \max\{X_i \colon 1 \leq i \leq n\}$

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Question: What are the possible (non-degenerate) limit distributions of appropriately normalized and centralized S_n and M_n?

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Consider first the limit distribution of S_n.

Question: What kind of non-degenerate r.v. Z are the limit distributions of $a_n^{-1}(S_n - b_n)$, for some sequences of reals $a_n > 0$ und $b_n \in \mathbb{R}$, $n \in \mathbb{N}$?

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Definition: A r.v. X is called *stable, (α-stable, Lévy-stable)*, iff for all $c_1, c_2 \in \mathbb{R}_+$ and the i.i.d. copies X_1 and X_2 of X, there exist constantes $a(c_1, c_2) \in \mathbb{R}$ and $b(c_1, c_2) \in \mathbb{R}$, such that $c_1 X_1 + c_2 X_2$ und $a(c_1, c_2)X + b(c_1, c_2)$ are identically distributed.

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Question: What are the possible (non-degenerate) limit distributions of appropriately normalized and centralized S_n and M_n?

Consider first the limit distribution of S_n.

Question: What kind of non-degenerate r.v. Z are the limit distributions of $a_n^{-1}(S_n - b_n)$, for some sequences of reals $a_n > 0$ und $b_n \in \mathbb{R}$, $n \in \mathbb{N}$?

Notation: $\lim_{n\to\infty} a_n^{-1}(S_n - b_n) \stackrel{\mathrm{d}}{=} Z$

Definition: A r.v. X is called *stable, (α-stable, Lévy-stable)*, iff for all $c_1, c_2 \in \mathbb{R}_+$ and the i.i.d. copies X_1 and X_2 of X, there exist constantes $a(c_1, c_2) \in \mathbb{R}$ and $b(c_1, c_2) \in \mathbb{R}$, such that $c_1 X_1 + c_2 X_2$ und $a(c_1, c_2)X + b(c_1, c_2)$ are identically distributed.
Notation: $c_1 X_1 + c_2 X_2 \stackrel{\mathrm{d}}{=} a(c_1, c_2)X + b(c_1, c_2)$

Theorem

The family of stable distributions coincides whith the limit distributions of appropriately normalized and centralized sums of i.i.d. r.v..

Proof e.g. in Rényi, 1962.

Stable distributions (contd.)

Theorem: The characteristic function of a stable distribution X is given as:

$$\varphi_X(t) = E[\exp\{iXt\}] = \exp\{i\gamma t - c|t|^\alpha(1 + i\beta \mathrm{signum}(t) z(t,\alpha))\}\,, \quad (4)$$

where $\gamma \in \mathbb{R}$, $c > 0$, $\alpha \in (0,2]$, $\beta \in [-1,1]$ and

$$z(t,\alpha) = \begin{cases} \tan(\frac{\pi\alpha}{2}) & \text{wenn } \alpha \neq 1 \\ -\frac{2}{\pi}\ln|t| & \text{wenn } \alpha = 1 \end{cases}$$

Proof: Lévy 1954, Gnedenko und Kolmogoroff 1960.

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Definition: The parameter α in (4) is called *the form parameter or characteristical exponent*, the corresponding distribution is called α-stable and its distribution function is denoted by G_α.

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Definition: The parameter α in (4) is called *the form parameter or characteristical exponent*, the corresponding distribution is called α-stable and its distribution function is denoted by G_α.

Definition: Let X be a r.v. with distribution function F. Assume that there exists two sequences of reals $a_n > 0$ and $b_n \in \mathbb{R}$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} a_n^{-1}(S_n - b_n) = G_\alpha$, for some α-stable distribution G_α. Then we say that "F belongs to the domain of attraction of G_α".
Notation: $F \in DA(G_\alpha)$.

Stable distributions (contd.)

Remark: $X \sim G_2 \Longleftrightarrow \varphi_X(t) = \exp\{i\gamma t - \frac{1}{2}t^2(2c)\} \Longleftrightarrow X \sim N(\gamma, 2c)$

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Exercise: Show that $F \in DA(G_2) \iff F \in DA(\phi)$, where ϕ is the standard normal distribution $N(0,1)$.
Hint: The Convergence to Types Theorem could be used.

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Definition: The r.v. Z and $\tilde{Z}$ *are of the same type* if there exist the constants $\sigma > 0$ and $\mu \in \mathbb{R}$, such that $\tilde{Z} \stackrel{\mathrm{d}}{=} (Z - \mu)/\sigma$, i.e. $\tilde{F}(x) = F(\mu + \sigma x)$, $\forall x \in \mathbb{R}$, where F and $\tilde{F}$ are the distribution functions of Z and $\tilde{Z}$, respecitvely.

The Convergence to Types Theorem

Let Z, $\tilde{Z}$, Y_n, $n \geq 1$, be not almost surely constant r.v.
Let $a_n, \tilde{a}_n, b_n, \tilde{b}_n \in \mathbb{R}$, $n \in \mathbb{N}$, be sequences of reals with $a_n, \tilde{a}_n > 0$.

(i) If

$$\lim_{n\to\infty} a_n^{-1}(Y_n - b_n) = Z \text{ and } \lim_{n\to\infty} \tilde{a}_n^{-1}(Y_n - \tilde{b}_n) = \tilde{Z} \tag{5}$$

then there exist $A > 0$ und $B \in \mathbb{R}$, such that

$$\lim_{n\to\infty} \frac{\tilde{a}_n}{a_n} = A \text{ and } \lim_{n\to\infty} \frac{\tilde{b}_n - b_n}{a_n} = B \tag{6}$$

and

$$\tilde{Z} \stackrel{\mathrm{d}}{=} (Z - B)/A. \tag{7}$$

(ii) Assume that (6) holds. Then each of the two relations in (5) implies the other and also (7) holds.

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Proof: See Resnick 1987, Prop. 0.2, Seite 7.