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     It is used frequently in portfolio theory.

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     Example
     $L_1 \sim N(0,2)$, $L_2 \sim t_4$ (Student's t-distribution with $m = 4$ degrees of freedom)
     $\sigma^2(L_1) = 2$ and $\sigma^2(L_2) = \frac{m}{m-2} = 2$ hold
     However the probability of losses is much larger for L_2 than for L_1.
     Plot the logarithm of the quotient $\ln[P(L_2 > x)/P(L_1 > x)]$!

2. **Value at Risk** ($VaR_\alpha(L)$)

   Definition: Let L be the loss distribution with distribution function F_L. Let $\alpha \in (0,1)$ be a given confindence level.

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If F is strictly monotone increasing, then $F^{-1} = F^{\leftarrow}$ holds.

Exercise: Compute $F^{\leftarrow}$ for $F\colon [0,+\infty) \to [0,1]$ with

$$F(x) = \begin{cases} 1/2 & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases}$$

Value at Risk (contd.)

Definition: Let $F\colon \mathbb{R} \to \mathbb{R}$ be a (monotone increasing) distribution function and $q_\alpha(F) := \inf\{x \in \mathbb{R}\colon F(x) \geq \alpha\}$ be α-quantile of F.

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Example: Let $L \sim N(\mu, \sigma^2)$.
Then $VaR_\alpha(L) = \mu + \sigma q_\alpha(\Phi) = \mu + \sigma\Phi^{-1}(\alpha)$ holds, where Φ is the distribution function of a random variable $X \sim N(0,1)$.

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Exercise: Consider a portfolio consisting of 5 pieces of an asset A. The today's price of A is $S_0 = 100$. The daily logarithmic returns are i.i.d.: $X_1 = \ln \frac{S_1}{S_0}$, $X_2 = \ln \frac{S_2}{S_1}, \ldots \sim N(0, 0.01)$. Let L_1 be the 1-day portfolio loss in the time interval (today, tomorrow).

(a) Compute $VaR_{0.99}(L_1)$.

(b) Compute $VaR_{0.99}(L_{100})$ and $VaR_{0.99}(L_{100}^{\Delta})$, where L_{100} is the 100-day portfolio loss over a horizon of 100 days starting with today. L_{100}^{Δ} is the linearization of the above mentioned 100-day PF-portfolio loss.

Hint: For $Z \sim N(0,1)$ use the equality $F_Z^{-1}(0.99) \approx 2.3$.

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$CVaR_\alpha(L) = E(L|L \geq VaR_\alpha(L)) = \frac{E(LI_{[q_\alpha(L),\infty)}(L))}{P(L \geq q_\alpha(L))} =$
$\frac{1}{1-\alpha} E(LI_{[q_\alpha(L),\infty)}) = \frac{1}{1-\alpha} \int_{q_\alpha(L)}^{+\infty} l dF_L(l)$

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If F_L **is discrete** the *generalized CVaR* is defined as follows:

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Then $CVaR_\alpha(L) = \frac{1}{1-\alpha} \int_\alpha^1 VaR_p(L) dp$ holds.

Conditional Value at Risk (contd.)
Example 1:

(a) Let $L \sim Exp(\lambda)$. Compute $CVaR_\alpha(L)$.

(b) Let the distribution function F_L of the loss function L be given as follows : $F_L(x) = 1 - (1 + \gamma x)^{-1/\gamma}$ for $x \geq 0$ and $\gamma \in (0, 1)$. Compute $CVaR_\alpha(L)$.

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Example 2:

Let $L \sim N(0, 1)$. Let ϕ und Φ be the density and the distribution function of L, respectively. Show that $CVaR_\alpha(L) = \frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ holds.
Let $L' \sim N(\mu, \sigma^2)$. Show that $CVaR_\alpha(L') = \mu + \sigma\frac{\phi(\Phi^{-1}(\alpha))}{1-\alpha}$ holds.

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Exercise:
Let the loss L be distributed according to the Student's t-distribution with $\nu > 1$ degrees of freedom. The density of L is

$$g_\nu(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\Gamma(\nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}$$

Show that $CVaR_\alpha(L) = \frac{g_\nu(t_\nu^{-1}(\alpha))}{1-\alpha} \left(\frac{\nu + (t_\nu^{-1}(a))^2}{\nu - 1}\right)$, where t_ν is the distribution function of L.

Methods for the computation of VaR und CVaR

Consider the portfolio value $V_m = f(t_m, Z_m)$, where Z_m is the vector of risk factors.

Let the loss function over the interval $[t_m, t_{m+1}]$ be given as $L_{m+1} = l_{[m]}(X_{m+1})$, where X_{m+1} is the vector of the risk factor changes, i.e.

$$X_{m+1} = Z_{m+1} - Z_m.$$

Consider observations (historical data) of risk factor values $Z_{m-n+1}, \ldots, Z_m$.
How to use these data to compute/estimate $VaR(L_{m+1})$, $CVaR(L_{m+1})$?

The empirical VaR and the empirical CVaR

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$$F_n(x) = \frac{1}{n} \sum_{k=1}^{n} I_{[x_k, +\infty)}(x)$$

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Assumption: $x_1 > x_2 > \ldots > x_n$. Then $q_\alpha(F_n) = x_{[n(1-\alpha)]+1}$ holds, where $[y] := \sup\{n \in \mathbb{N} \colon n \leq y\}$ for every $y \in \mathbb{R}$.

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Lemma

Let $\hat{q}_\alpha(F) := q_\alpha(F_n)$ and let F be a strictly increasing function. Then $\lim_{n\to\infty} \hat{q}_\alpha(F) = q_\alpha(F)$ holds $\forall \alpha \in (0,1)$, i.e. the estimator $\hat{q}_\alpha(F)$ is consistent.

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The empirical estimator of CVaR is $\widehat{\mathrm{CVaR}}_\alpha(F) = \frac{\sum_{k=1}^{[n(1-\alpha)]+1} x_k}{[(n(1-\alpha)]+1}$

A non-parametric bootstrapping approach to compute the confidence interval of the estimator

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Let $\hat{\theta}(x_1, \ldots, x_n)$ be an estimator of θ, e.g. $\hat{\theta}(x_1, \ldots, x_n) = x_{[(n(1-\alpha)]+1}$ $\theta = q_\alpha(F)$.
The required confidence interval is an (a, b) with $a = a(x_1, \ldots, x_n)$ u. $b = b(x_1, \ldots, x_n)$, such that $P(a < \theta < b) = p$, for a given confidence level p.

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Case I: F is known.
Generate N samples $\tilde{x}_1^{(i)}, \tilde{x}_2^{(i)}, \ldots, \tilde{x}_n^{(i)}$, $1 \le i \le N$, by simulation from F (N should be large)

Let $\tilde{\theta}_i = \hat{\theta}\left(\tilde{x}_1^{(i)}, \tilde{x}_2^{(i)}, \ldots, \tilde{x}_n^{(i)}\right)$, $1 \le i \le N$.

Case I (cont.)

The empirical distribution function of $\hat{\theta}(x_1, x_2, \ldots, x_n)$ is given as

$$F_N^{\hat{\theta}} := \frac{1}{N} \sum_{i=1}^{N} I_{[\tilde{\theta}_i, \infty)}$$

and it tends to $F^{\hat{\theta}}$ for $N \to \infty$.

The required conficence interval is given as

$$\left(q_{\frac{1-p}{2}}(F_N^{\hat{\theta}}), q_{\frac{1+p}{2}}(F_N^{\hat{\theta}}) \right)$$

(assuming that the sample sizes N und n are large enough).

Case II: F is not known $\Rightarrow$ Bootstrapping!

The empirical distribution function of X_i, $1 \leq i \leq n$, is given as

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} I_{[x_i, \infty)}(x).$$

For n large $F_n \approx F$ holds.

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Generate samples from F_n be choosing n elementes in $\{x_1, x_2, \ldots, x_n\}$ and putting every element back to the set immediately after its choice

Assume N such samples are generated: $x_1^{*(i)}, x_2^{*(i)}, \ldots, x_n^{*(i)}$, $1 \leq i \leq N$.

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Compute $\theta_i^* = \hat{\theta}\left(x_1^{*(i)}, x_2^{*(i)}, \ldots, x_n^{*(i)}\right)$.

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A confidence interval (a, b) with confidence level p is given by

$$a = q_{(1-p)/2}(F_N^{\theta^*}),\ b = q_{(1+p)/2}(F_N^{\theta^*}).$$

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Compute $\theta_i^* = \hat{\theta}\left(x_1^{*(i)}, x_2^{*(i)}, \ldots, x_n^{*(i)}\right)$.

The empirical distribution of θ_i^* is given as $F_N^{\theta^*}(x) = \frac{1}{N} \sum_{i=1}^{N} I_{[\theta_i^*,\infty)}(x)$; it approximates the distribution function $F^{\hat{\theta}}$ of $\hat{\theta}(X_1, X_2, \ldots, X_n)$ for $N \to \infty$.

A confidence interval (a, b) with confidence level p is given by

$$a = q_{(1-p)/2}(F_N^{\theta^*}), \; b = q_{(1+p)/2}(F_N^{\theta^*}).$$

Thus $a = \theta^*_{[N(1+p)/2]+1}$, $b = \theta^*_{[N(1-p)/2]+1}$, where $\theta_1^* \geq \ldots \geq \theta_N^*$ is the sorted θ^* sample.

Summary of the non-parametric bootstrapping approach to compute confidence intervals

Input: Sample $x_1, x_2, \dots, x_n$ of the i.i.d. random variables $X_1, X_2, \dots, X_n$ with distribution function F and an estimator $\hat{\theta}(x_1, x_2, \dots, x_n)$ of an unknown parameter $\theta(F)$, A confidence level $p \in (0,1)$.

Output: A confidence interval I_p for θ with confidence level p.

- Generate N new Samples $x_1^{*(i)}, x_2^{*(i)}, \dots, x_n^{*(i)}$, $1 \le i \le N$, by chosing elements in $\{x_1, x_2, \dots, x_n\}$ and putting them back right after the choice.
- Compute $\theta_i^* = \hat{\theta}\left(x_1^{*(i)}, x_2^{*(i)}, \dots, x_n^{*(i)}\right)$.
- Setz $I_p := \left(\theta^*_{[N(1+p)/2]+1,N}, \theta^*_{[N(1-p)/2]+1,N}\right)$, where $\theta^*_{1,N} \ge \theta^*_{2,N} \ge \dots \theta^*_{N,N}$ is obtained by sorting $\theta_1^*, \theta_2^*, \dots, \theta_N^*$.

An approximative solution without bootstrapping

Input: A sample $x_1, x_2, \ldots, x_n$ of the random variables X_i, $1 \leq i \leq n$, i.i.d. with unknown continuous distribution function F, a confidence level $p \in (0, 1)$.

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Output: A $p' \in (0,1)$, with $p \leq p' \leq p + \epsilon$, for some small ϵ, and a confidence interval (a, b) for $q_\alpha(F)$, i.e. $a = a(x_1, x_2, \ldots, x_n)$, $b = b(x_1, x_2, \ldots, x_n)$, such that

$P(a < q_\alpha(F) < b) = p'$ and $P(a \geq q_\alpha(F)) = P(b \leq q_\alpha(F) \leq (1-p)/2$ holds.

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Assume w.l.o.g. that the sample is sorted $x_1 \geq x_2 \geq \ldots \geq x_n$.

Determine $i > j$, $i, j \in \{1, 2, \ldots, n\}$, and the smallest $p' > p$, such that

$$P\left(x_i < q_\alpha(F) < x_j\right) = p' \qquad (*) \qquad \text{and}$$

$$P\left(x_i \geq q_\alpha(F)\right) \leq (1-p)/2 \text{ and } P\left(x_j \leq q_\alpha(F)\right) \leq (1-p)/2 (**).$$

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Set $b := x_j$ and $a := x_i$.

Possibilities to generate a sample of losses $x_1, \ldots, x_n$

(i) Historical simulation

Let $x_{m-n+1}, \ldots, x_m$ be historical observations of the risk factor changes $X_{m-n+1}, \ldots, X_m$; the historically realized losses are given as $l_k = l_{[m]}(x_{m-k+1})$, $k = 1, 2, \ldots, n$,

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Analogously, we can consider the loss aggregated over a given time interval (number of days or general time units).
VaR and CVaR of the loss aggregated over a number of days, e.g. 10 days, over the days $m-n+10(k-1)+1, m-n+10(k-1)+2, \ldots, m-n+10(k-1)+10$, denoted by $l_k^{(10)}$ is given as

$$l_k^{(10)} = l_{[m]}\left(\sum_{j=1}^{10} x_{m-n+10(k-1)+j}\right) \qquad k = 1,\ldots,[n/10]$$

Historical simulation (contd.)

Advantages:

- simple implementation
- considers intrinsically the dependencies between the elements of the vector of the risk factors changes $X_{m-k} = (X_{m-k,1}, \dots, X_{m-k,d})$.

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Disadvantages:

- lots of historical data needed to get good estimators
- the estimated loss cannot be larger than the maximal loss experienced in the past

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where $V := V_m$, $w_i := w_{m,i}$, $w = (w_1, \ldots, w_d)^T$,
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Estimator for VaR: $\widehat{VaR}(L_{m+1}) = -Vw^T\hat{\mu} + V\sqrt{w^T\hat{\Sigma}w}\,\phi^{-1}(\alpha)$