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**Simplified case:** $Y_i$ are independent for $i = 1, 2, \ldots, m$.
Let $\Omega = \{0, 1\}^m$ be the state space of the random vector $Y$.
Consider the probability measure $P$ in $\Omega$:

$$P(\{y\}) = \prod_{i=1}^{m} \bar{p}_i^{y_i} (1 - \bar{p}_i)^{1 - y_i}, \; y \in \{0, 1\}^m.$$

The moment generating function of $L$ is $M_L(t) = \prod_{i=1}^{m} (e^{t e_i} \bar{p}_i + 1 - \bar{p}_i)$.

## IS in the case of Bernoulli mixture models (contd.)

Consider a probability measure $Q_t$:

$$Q_t(\{y\}) = \prod_{i=1}^{n} \left( \frac{\exp\{te_i y_i\}}{\exp\{te_i\}\bar{p}_i + 1 - \bar{p}_i} \bar{p}_i^{y_i} (1 - \bar{p}_i)^{1-y_i} \right).$$

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Let $\bar{q}_{t,i}$ be new default probabilities

$$\bar{q}_{t,i} := \exp\{te_i\}\bar{p}_i / (\exp\{te_i\}\bar{p}_i + 1 - \bar{p}_i).$$

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$\lim_{t\to\infty} \bar{q}_{t,i} = 1$ and $\lim_{t\to-\infty} \bar{q}_{t,i} = 0$ imply that $E^{Q_t}(L)$ takes all values in $(0, \sum_{i=1}^{m} e_i)$ for $t \in \mathbb{R}$.

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Choose $t$, such that $\sum_{i=1}^{m} e_i \bar{q}_{t,i} = c$.

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**Algorithm:** IS for the conditional loss distribution

(1) For a given $z$ compute the conditional default probabilities $p_i(z)$ (as in the simplified case) and solve the equation

$$\sum_{i=1}^{m} e_i \frac{\exp\{te_i\} p_i(z)}{\exp\{te_i\} p_i(z) + 1 - p_i(z)} = c \,.$$

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The solution $t_z$ (which clearly depends also on the given $c$) specifies the correct *degree of tilting*.

(2) Generate $n_1$ conditional realisations of the vector of default indicators $(Y_1, \ldots, Y_m)$, $Y_i$ are simulated from *Bernoulli*$(q_i)$, $i = 1, 2, \ldots, m$, with

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(3) Let $M_L^{(z)}(t) := \prod[\exp\{t_z e_i\}p_i(z) + 1 - p_i(z)]$ be the conditional moment generating function of $L$. Let $L^{(1)}$, $L^{(2)}$,...,$L^{(n_1)}$ be the $n_1$ conditional realisations of $L$ for the $n_1$ simulated realisations of $Y_1, Y_2, \ldots, Y_m$. Compute the *IS*-estimator for the tail probability of the conditional loss distribution:

$$\hat{\theta}_{n_1}^{(IS)}(z) = M_L^{(z)}(t)\frac{1}{n_1}\sum_{j=1}^{n_1} I_{L^{(j)} \geq c} \exp\{-t_z L^{(j)}\}L^{(j)}.$$

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Naive approach: Generate many realisations $z$ of the impact factors $Z$ and compute $\hat{\theta}_{n_1}^{(IS)}(z)$ for every one of them. The required estimator is the average of $\hat{\theta}_{n_1}^{(IS)}(z)$ over all realisations $z$.
This is not the most efficient approach, see Glasserman and Li (2003).

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A better alternative: IS for the impact factors.

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Assumption: $Z \sim N_k(0, \Sigma)$ (e.g. like in the probit-normal or logit BMD), where $k \in \mathbb{N}$ is the number of factors.

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Let the IS density $g$ be the density of $N_k(\mu, \Sigma)$ for a new expected vector $\mu \in \mathbb{R}^k$. A good choice of $\mu$ should lead to frequent realisations of $z$ which imply high conditional default probabilities $p_i(z)$.

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The likelihood ratio:

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(3) compute the IS estimator for the independent excess probability:

$$\hat{\theta}^{(IS)}_n = \frac{1}{n} \sum_{i=1}^{n} r_\mu(z_i) \hat{\theta}^{(IS)}_{n_1}(z_i)$$

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Glasserman und Li (2003) propose some numerical solution approaches.