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If all function f_i coincide, i.e. $f_i = f$, $\forall i$, we get $N|Z \sim \text{Bin}(n, f(Z))$ for the number $N = \sum_{i=1}^n X_i$ of defaults.

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If $\lambda_i(Z) << 1$ we get for the number $\tilde{N} = \sum_{i=1}^n \bar{X}_i \approx \sum_{i=1}^n X_i$ of defaults:

$$\tilde{N}|Z \sim Poisson(\bar{\lambda}(Z)), \text{ where } \bar{\lambda} = \sum_{i=1}^n \lambda_i(Z).$$

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The goal: approximate the loss disribution by a discrete distribution and derive the generator function for the latter.

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The loss function is then given by $L = \sum_{i=1}^n \bar{X}_i v_i L_0 \approx \sum_{i=1}^n X_i v_i L_0$, where $\bar{X}_i$ is the loss indicator and $(X_1, \ldots, X_n)$ has a PMD with factor vector $(Z_1, Z_2, \ldots, Z_m)$ as described above.

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$X_i|Z \sim Poi(\lambda_i(Z))$, $\forall i \Longrightarrow g_{X_i|Z}(t) = \exp\{\lambda_i(Z)(t-1)\}$, $\forall i \Longrightarrow$
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with $\mu := \sum_{i=1}^n \lambda_i(Z) = \sum_{i=1}^n \left(\bar{\lambda}_i \sum_{j=1}^m a_{ij} Z_j\right)$.

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Analogous computations as in the case of $g_N(t)$ yield:

$$g_L(t) = \prod_{j=1}^{m} \left(\frac{1-\delta_j}{1-\delta_j \Lambda_j(t)} \right)^{\alpha_j} \text{ wobei } \Lambda_j(t) = \frac{1}{\mu_j} \sum_{i=1}^{n} \bar{\lambda}_i a_{ij} t^{v_i}.$$

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Assume that $\bar{\lambda}_i = \bar{\lambda} = 0.15$, for $i = 1, 2, \ldots, n$, $\alpha_j = \alpha = 1$, $\beta_j = \beta = 1$, $a_{i,j} = 1/m$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

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$g_N^{(k)}(0) = \sum_{l=0}^{k-1} \binom{k-1}{l} g_N^{(k-1-l)}(0) \sum_{j=1}^{m} l! \alpha_j \delta_j^{l+1}$, where $k > 1$.

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$$\widehat{CVaR}_\alpha^{(MC)}(L) = \frac{1}{\sum_{i=1}^{n} I_{(q_\alpha,+\infty)}(L^{(i)})} \sum_{i=1}^{n} L^{(i)} I_{(q_\alpha,+\infty)}(L^{(i)}),$$

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$\widehat{CVaR}_\alpha^{(MC)}(L)$ is unstable, i.e. it has a very high variance unless the number of simulation runs is very high.

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Examples:
Set $h(x) = I_A(x)$ to compute the probability of an event A.

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Basics of importance sampling

Let X be a r.v. in a probability space $(\Omega, \mathcal{F}, P)$ with absolutely continuous distribution function and density function f.

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In case of rare events, e.g. $h(x) = I_A(x)$ with $\mathbb{P}(A) << 1$, the convergence is very slow.

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Goal: choose an IS density g such that the variance of the IS estimator is much smaller than the variance of the standard MC-estimator.

$$var\left(\hat{\theta}_n^{(IS)}\right) = \frac{1}{n}(E_g(h^2(X)r^2(X)) - \theta^2)$$

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Goal: choose g such that $E_g(h^2(X)r^2(X))$ becomes small, i.e. such that $r(x)$ is small for $x \geq c$. Aquivalently, the event $X \geq c$ should be more probable under density g than under density f.

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(A unique solution of the above equality exists for all relevant values of c, see e.g. Embrechts et al. for a proof).

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The IS algorithm does not change: Simulate independent realisations of X_i in $(\Omega, \mathcal{F}, Q_t)$ and set $\hat{\theta}_n^{(IS)} = (1/n)\sum_{i=1}^n X_i r_t(X_i)$.