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Standard empirical estimator of Kendalls Tau:
$(\widehat{\rho_\tau})_{ij} = \binom{n}{2}^{-1} \sum_{1 \le k < l \le n} sign((X_{k,i} - X_{l,i})(X_{k,j} - X_{l,j}))$.

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Eigenvalue approach (Rousseeuw and Molenberghs 1993)

- Compute the spectral decomposition $\hat{R} = \Gamma\Lambda\Gamma^T$ of $\hat{R}$, where Λ is a diagonal matrix, containing the eigenvalues of $\hat{R}$ on the diagonal, and Γ is an orthogonal matrix with the eigenvectors of $\hat{R}$ in its columns.

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- Set $R^* := D\tilde{R}D$ where D is a diagonal matrix with $D_{k,k} = 1/\sqrt{\tilde{R}_{k,k}}$.

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for $k = 1, 2, \ldots, n$ (see Genest und Rivest 1993).

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- a non-parametric estimation method;
  $\hat{F}_i$ is the empirical distribution function $\hat{F}_i(x) = \frac{1}{n+1}\sum_{t=1}^{n} I_{\{X_{t,i} \leq x\}}$, $1 \leq i \leq d$.

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where $g_{\xi,R}$ is the cumulative density function of a d-dimensional standard t-distribution with ξ degrees of freedom and correlation matrix R, and g_{ξ} is the density function of a univariate standard t-distribution with ξ degrees of freedom.