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Definition: Let $\phi\colon [0,1] \to [0,+\infty]$ be a continuous, strictly monotone decreasing function with $\phi(1) = 0$. The pseudo-inverse function $\phi^{[-1]}\colon [0,\infty] \to [0,1]$ of ϕ is defined by

$$\phi^{[-1]}(t) = \begin{cases} \phi^{-1}(t) & 0 \le t \le \phi(0) \\ 0 & \phi(0) \le t \le \infty \end{cases}$$

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$\phi^{[-1]}$ is continuous and monotone decreasing on $[0,\infty]$, strictly monotone decreasing on $[0,\phi(0)]$ and $\phi^{[-1]}(\phi(u)) = u$ for $u \in [0,1]$ holds. Moreover

$$\phi(\phi^{[-1]}(t) = \begin{cases} t & 0 \le t \le \phi(0) \\ \phi(0) & \phi(0) \le t \le +\infty \end{cases}$$

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If $\phi(0) = +\infty$, then $\phi^{[-1]} = \phi^{-1}$.

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Examples: Gumbel Copulas: $\phi(t) = (-\ln t)^{\theta}$, $\theta \geq 1$, $t \in [0,1]$. Then $C_{\theta}^{Gu}(u_1,u_2) = \phi^{[-1]}(\phi(u_1) + \phi(u_2)) = \exp\left(-[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}]^{1/\theta}\right)$ is the Gumbel copula with parameter θ.

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Theorem: Let $(X_1,X_2)^T$ be a random vector with continuous marginal distributions and an Archimedian copula C generated by ϕ. Then $\rho_\tau(X_1,X_2) = 1 + 4\int_0^1 \frac{\phi(t)}{\phi'(t)} dt$ holds.

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Theorem: (Kimberling 1974)
Let $\phi\colon [0,1] \to [0,\infty]$ be a continuous, strictly monotone decreasing function with $\phi(0) = \infty$ and $\phi(1) = 0$. The function $C\colon [0,1]^d \to [0,1]$, $C(u) := \phi^{-1}(\phi(u_1) + \phi(u_2) + \ldots + \phi(u_d))$ is a copula for $d \geq 2$ iff ϕ^{-1} is completely monotone on $[0,\infty)$.

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Lemma: A function $\psi\colon [0,\infty) \to [0,\infty)$ is completely monotone with $\psi(0) = 1$ iff ψ is the Laplace-Stieltjes transform of some distribution function G on $[0,\infty)$, i.e. $\psi(s) = \int_0^\infty e^{-sx} dG(x)$, $s \geq 0$.

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Then

$$Prob(U_1 \leq u_1, U_2 \leq u_2, \ldots, U_d \leq u_d) = \psi(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \ldots + \psi^{-1}(u_d))$$

and the distribution function of $U = (U_1, U_2, \ldots, U_d)$ is an Archimedian copula with generator ψ^{-1}.

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$$Prob(U_1 \leq u_1, U_2 \leq u_2, \ldots, U_d \leq u_d) = \psi(\psi^{-1}(u_1) + \psi^{-1}(u_2) + \ldots + \psi^{-1}(u_d))$$

and the distribution function of $U = (U_1, U_2, \ldots, U_d)$ is an Archimedian copula with generator ψ^{-1}.

Advantages and disadvantages of Archimedian copulas:

- can model a broader class of dependencies

Multivariate Archimedian copulas (contd.)

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Multivariate Archimedian copulas (contd.)

Theorem: Let G be a distribution function on $[0,\infty)$ such that $G(0)=0$. Let ψ be the Laplace-Stieltjes transform of G, i.e. $\psi(s)=\int_0^\infty e^{-sx}dG(x)$ for $s\geq 0$. Let X be a r.v. with distribution function G and let $U_1, U_2, \ldots, U_d$ be conditionally independent r.v. for $X=x$, $x\in \mathbb{R}^+$, with conditional distribution function $F_{U_k|X=x}(u)=\exp(-x\psi^{-1}(u))$ for $u\in[0,1]$.
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Advantages and disadvantages of Archimedian copulas:

- can model a broader class of dependencies
- have a closed form representation
- depend on a small number of parameters in general
- the generator function needs to fulfill quite restrictive technical assumptions

Simulation of Gaussian copulas

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Observe: Consider a symmetric positive definite matrix $R \in \mathbb{R}^{d \times d}$ and its Cholesky factorization $AA^T = R$ with $A \in \mathbb{R}^{d \times d}$. If $Z_1, Z_2, \dots, Z_d \sim N(0, 1)$ are independent, then $\mu + AZ \sim N_d(\mu, R)$.

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Algorithm: for the generation of a random vector $U = (U_1, U_2, \ldots, U_d)$ whose distribution function is the copula C_R^{Ga}, R positive definite with all ones on the main diagonal.

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- Set $U_k := \phi(X_k)$ for $k = 1, 2, \ldots, d$, where ϕ is the standard normal distribution function.
- Output $U = (U_1, U_2, \ldots, U_d)$; U has distribution function C_R^{Ga}.

Simulation of t-copulas

Simulation of t-copulas

Algorithm: for the generation of a random vector $U = (U_1, U_2, \ldots, U_d)$ whose distribution function is the copula $C^t_{\nu,R}$, R positive definite with all ones on the main diagonal, $\nu \in \mathbb{N}$.

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- Output $U = (U_1, U_2, \dots, U_d)$; $U = (U_1, U_2, \dots, U_d)$ has distribution function $C^t_{\nu,R}$.

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A generic algorithm to generate a random vector $U = (U_1, U_2, \ldots, U_d)$ with a given Archimedean copula as cumulative distribution function

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The generator $\varphi(t) = (t^{-\theta} - 1)/\theta$, $\theta > 0$ yields the Clayton copula C_θ^{Cl}.

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For $X \sim Gamma(1/\theta, 1)$ with d.f. $f_X(x) = \left(x^{1/\theta-1}e^{-x}\right)/\Gamma(1/\theta)$ we have:
$E(e^{-sX}) = \int_0^\infty e^{-sx}\frac{1}{\Gamma(1/\theta)}x^{1/\theta-1}e^{-x}dx = (s+1)^{-1/\theta} = \tilde{\varphi}^{-1}(s)$.

Simulation of the Clayton copula ($\theta > 0$)

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Let X be a positive stable r.v., $X \sim St(1/\theta, 1, \gamma, 0)$ with $\gamma = (\cos(\pi/(2\theta)))^{\theta} > 0$ (and $\alpha = \frac{1}{\theta}$, $\beta = 1$, $\delta = 0$)

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The Laplace-Stieltjes transform of F_X is $\varphi(t) = \exp(-t^{1/\theta})$, the inverse of the generator of the Gumbel copula C_θ^{Gu}.

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The simulation of $Z \sim ST(\alpha, \beta, 1, 0)$ is not straightforward (no analytic expression for the inverse $F^{-1}(x)$ nor $F(x)$; see also Nolan 2002).

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For $\alpha \neq 1$ we get: $X = \delta + \gamma Z \sim St(\alpha, \beta, \gamma, \delta)$.

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The case $\alpha = 1$ is more complicated.

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Alternative approach:
Let $\theta \geq 1$ and $\bar{F}(x) = 1 - F(x) := \exp(-x^{1/\theta})$ for $x \geq 0$.

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Let X be a positive stable r.v., $X \sim St(1/\theta, 1, \gamma, 0)$ with $\gamma = (\cos(\pi/(2\theta)))^\theta > 0$ (and $\alpha = \frac{1}{\theta}$, $\beta = 1$, $\delta = 0$)
The Laplace-Stieltjes transform of F_X is $\varphi(t) = \exp(-t^{1/\theta})$, the inverse of the generator of the Gumbel copula C_θ^{Gu}.

The simulation of $Z \sim ST(\alpha, \beta, 1, 0)$ is not straightforward (no analytic expression for the inverse $F^{-1}(x)$ nor $F(x)$; see also Nolan 2002).

For $\alpha \neq 1$ we get: $X = \delta + \gamma Z \sim St(\alpha, \beta, \gamma, \delta)$.

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The distribution function of $(\bar{F}(Z_1), \bar{F}(Z_2))^T$ is C_θ^{Gu}. Convince yourself!

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Question 2: Estimation of the parameters of the prespecified family of copulas used for the modelling?