Tail dependency and survival copulas

Tail dependency and survival copulas

Definition: Let the copula C be the c.d.f. of a random vector $(U_1, U_2, \ldots, U_d)$ with $U_i \sim U[0,1]$, $i = 1, 2, \ldots, d$. The c.d.f. of $(1 - U_1, 1 - U_2, \ldots, 1 - U_d)$ is called *survival copula* of C and is denoted by $\hat{C}$.

Tail dependency and survival copulas

Definition: Let the copula C be the c.d.f. of a random vector $(U_1, U_2, \ldots, U_d)$ with $U_i \sim U[0,1]$, $i = 1, 2, \ldots, d$. The c.d.f. of $(1 - U_1, 1 - U_2, \ldots, 1 - U_d)$ is called *survival copula* of C and is denoted by $\hat{C}$.

Lemma: Let X be a random vector with multivariate tail distribution function $\bar{F}$ ($\bar{F}(x_1, x_2, \ldots, x_d) := Prob(X_1 > x_1, X_2 > x_2, \ldots, X_d > x_d)$) and marginal distributions F_i, $i = 1, 2, \ldots, d$. Let $\bar{F}_i := 1 - F_i$, $i = 1, 2, \ldots, d$. Then the following holds

$$\bar{F}(x_1, x_2, \ldots, x_d) = \hat{C}(\bar{F}_1(x_1), \bar{F}_2(x_2), \ldots, \bar{F}_d(x_d)).$$

Tail dependency and survival copulas

Definition: Let the copula C be the c.d.f. of a random vector $(U_1, U_2, \ldots, U_d)$ with $U_i \sim U[0,1]$, $i = 1, 2, \ldots, d$. The c.d.f. of $(1 - U_1, 1 - U_2, \ldots, 1 - U_d)$ is called *survival copula* of C and is denoted by $\hat{C}$.

Lemma: Let X be a random vector with multivariate tail distribution function $\bar{F}$ ($\bar{F}(x_1, x_2, \ldots, x_d) := Prob(X_1 > x_1, X_2 > x_2, \ldots, X_d > x_d)$) and marginal distributions F_i, $i = 1, 2, \ldots, d$. Let $\bar{F}_i := 1 - F_i$, $i = 1, 2, \ldots, d$. Then the following holds

$$\bar{F}(x_1, x_2, \ldots, x_d) = \hat{C}(\bar{F}_1(x_1), \bar{F}_2(x_2), \ldots, \bar{F}_d(x_d)).$$

Lemma: For any copula C and its survival copula $\hat{C}$ the following holds $\hat{C}(1 - u_1, 1 - u_2) = 1 - u_1 - u_2 + C(u_1, u_2)$.

Tail dependency and survival copulas

Definition: Let the copula C be the c.d.f. of a random vector $(U_1, U_2, \ldots, U_d)$ with $U_i \sim U[0,1]$, $i = 1, 2, \ldots, d$. The c.d.f. of $(1 - U_1, 1 - U_2, \ldots, 1 - U_d)$ is called *survival copula* of C and is denoted by $\hat{C}$.

Lemma: Let X be a random vector with multivariate tail distribution function $\bar{F}$ ($\bar{F}(x_1, x_2, \ldots, x_d) := Prob(X_1 > x_1, X_2 > x_2, \ldots, X_d > x_d)$) and marginal distributions F_i, $i = 1, 2, \ldots, d$. Let $\bar{F}_i := 1 - F_i$, $i = 1, 2, \ldots, d$. Then the following holds

$$\bar{F}(x_1, x_2, \ldots, x_d) = \hat{C}(\bar{F}_1(x_1), \bar{F}_2(x_2), \ldots, \bar{F}_d(x_d)).$$

Lemma: For any copula C and its survival copula $\hat{C}$ the following holds $\hat{C}(1 - u_1, 1 - u_2) = 1 - u_1 - u_2 + C(u_1, u_2)$.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and a unique copula C. The following equalities hold $\lambda_U(X_1, X_2) = \lim_{u \to 1^-} \frac{1-2u+C(u,u)}{1-u}$ and $\lambda_L(X_1, X_2) = \lim_{u \to 0^+} \frac{C(u,u)}{u}$, provided that the limits exist.

Exmaples of copulas:

Exmaples of copulas:

The Gumbel family of copulas:

$$C_\theta^{\mathrm{Gu}}(u_1, u_2) = \exp\left(-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta\right]^{1/\theta}\right),\ \theta \geq 1$$

We have $\lambda_U = 2 - 2^{1/\theta}$, $\lambda_L = 0$.

Exmaples of copulas:

The Gumbel family of copulas:

$$C_\theta^{\text{Gu}}(u_1, u_2) = \exp\left(-\left[(-\ln u_1)^\theta + (-\ln u_2)^\theta\right]^{1/\theta}\right), \ \theta \geq 1$$

We have $\lambda_U = 2 - 2^{1/\theta}$, $\lambda_L = 0$.

The Clayton family of copulas:

$$C_\theta^{\text{Cl}}(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{1/\theta}, \ \theta > 0$$

We have $\lambda_U = 0$, $\lambda_L = 2^{-1/\theta}$.

Elliptical copulas

Elliptical copulas

Definition: Let X be a d-dimensional random vector. Let $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d \times d}$ be constants, and let $\psi \colon [0, \infty) \to \mathbb{R}$ be a function such that $\phi_{X-\mu} = \psi(t^T \Sigma t)$ holds for the characteristic function $\phi_{X-\mu}$ of $X - \mu$. Then X is an elliptically distributed random vector with parameters μ, Σ, ψ. Notation: $X \sim E_d(\mu, \Sigma, \psi)$.

Elliptical copulas

Definition: Let X be a d-dimensional random vector. Let $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d\times d}$ be constants, and let $\psi\colon [0,\infty) \to \mathbb{R}$ be a function such that $\phi_{X-\mu} = \psi(t^T \Sigma t)$ holds for the characteristic function $\phi_{X-\mu}$ of $X - \mu$. Then X is an elliptically distributed random vector with parameters μ, Σ, ψ. Notation: $X \sim E_d(\mu, \Sigma, \psi)$.

ψ is the *generating function* (or the *generator*) of X.

Elliptical copulas

Definition: Let X be a d-dimensional random vector. Let $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d \times d}$ be constants, and let $\psi \colon [0, \infty) \to \mathbb{R}$ be a function such that $\phi_{X-\mu} = \psi(t^T \Sigma t)$ holds for the characteristic function $\phi_{X-\mu}$ of $X - \mu$. Then X is an elliptically distributed random vector with parameters μ, Σ, ψ. Notation: $X \sim E_d(\mu, \Sigma, \psi)$.

ψ is the *generating function* (or the *generator*) of X.

For $d = 1$ the elliptical distributions coincide with distributions which are symmetric w.r.t. μ. (*Convince yourself! Exploit the stochastic representation of elliptical distributions.*)

Elliptical copulas

Definition: Let X be a d-dimensional random vector. Let $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d\times d}$ be constants, and let $\psi\colon [0,\infty) \to \mathbb{R}$ be a function such that $\phi_{X-\mu} = \psi(t^T \Sigma t)$ holds for the characteristic function $\phi_{X-\mu}$ of $X - \mu$. Then X is an elliptically distributed random vector with parameters μ, Σ, ψ. Notation: $X \sim E_d(\mu, \Sigma, \psi)$.

ψ is the *generating function* (or the *generator*) of X.

For $d = 1$ the elliptical distributions coincide with distributions which are symmetric w.r.t. μ. (*Convince yourself! Exploit the stochastic representation of elliptical distributions.*)

Theorem:(Stochastic representation)
A d-dimensional random vector X is elliptically distributed, $X \sim E_d(\mu, \Sigma, \psi)$ with *rang*$(\Sigma) = k$, iff there exist a matrix $A \in \mathbb{R}^{d\times k}$, $A^T A = \Sigma$, a nonnegative r.v. R and a k-dimensional random vector U uniformly distributed on the unit ball $\mathcal{S}^{k-1} = \{z \in \mathbb{R}^k\colon z^T z = 1\}$, such that R and U are independent and $X \stackrel{d}{=} \mu + RAU$.

Elliptical copulas

Definition: Let X be a d-dimensional random vector. Let $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d \times d}$ be constants, and let $\psi \colon [0, \infty) \to \mathbb{R}$ be a function such that $\phi_{X-\mu} = \psi(t^T \Sigma t)$ holds for the characteristic function $\phi_{X-\mu}$ of $X - \mu$. Then X is an elliptically distributed random vector with parameters μ, Σ, ψ. Notation: $X \sim E_d(\mu, \Sigma, \psi)$.

ψ is the *generating function* (or the *generator*) of X.

For $d = 1$ the elliptical distributions coincide with distributions which are symmetric w.r.t. μ. (*Convince yourself! Exploit the stochastic representation of elliptical distributions.*)

Theorem:(Stochastic representation)
A d-dimensional random vector X is elliptically distributed, $X \sim E_d(\mu, \Sigma, \psi)$ with *rang*$(\Sigma) = k$, iff there exist a matrix $A \in \mathbb{R}^{d \times k}$, $A^T A = \Sigma$, a nonnegative r.v. R and a k-dimensional random vector U uniformly distributed on the unit ball $\mathcal{S}^{k-1} = \{z \in \mathbb{R}^k \colon z^T z = 1\}$, such that R and U are independent and $X \stackrel{d}{=} \mu + RAU$.

Remark: An elliptically distributed random vector X ist *radial symmetric*, i.e. $X - \mu \stackrel{d}{=} \mu - X$.

Elliptical copulas (contd.)

Elliptical copulas (contd.)

Definition: Let $X \sim E_d(\mu, \Sigma, \psi)$ be an elliptically distributed random vector with c.d.f. F and marginal distributions $F_1, F_2, \ldots, F_d$. The unique copula C of X (or F) with $C(u) = F(F_1^{\leftarrow}(u_1), \ldots, F_d^{\leftarrow}(u_d))$, is called an *elliptical copula*.

Elliptical copulas (contd.)

Definition: Let $X \sim E_d(\mu, \Sigma, \psi)$ be an elliptically distributed random vector with c.d.f. F and marginal distributions $F_1, F_2, \ldots, F_d$. The unique copula C of X (or F) with $C(u) = F(F_1^{\leftarrow}(u_1), \ldots, F_d^{\leftarrow}(u_d))$, is called an *elliptical copula*.

Example: Gaussian copulas are elliptical copulas

Elliptical copulas (contd.)

Definition: Let $X \sim E_d(\mu, \Sigma, \psi)$ be an elliptically distributed random vector with c.d.f. F and marginal distributions $F_1, F_2, \ldots, F_d$. The unique copula C of X (or F) with $C(u) = F(F_1^{\leftarrow}(u_1), \ldots, F_d^{\leftarrow}(u_d))$, is called an *elliptical copula*.

Example: Gaussian copulas are elliptical copulas
Let C_R^{Ga} be the copula of a d-dimensional normal distribution with correlation matrix R. Then $C_R^{Ga}(u) = \phi_R^d(\phi^{-1}(u_1), \ldots, \phi^{-1}(u_d))$ holds, where ϕ_R^d is the c.d.f. of a d-dimensional normal distribution with expected vector 0 and correlation matrix R, and ϕ^{-1} is the inverse of the standard normal distribution function.

Elliptical copulas (contd.)

Definition: Let $X \sim E_d(\mu, \Sigma, \psi)$ be an elliptically distributed random vector with c.d.f. F and marginal distributions $F_1, F_2, \ldots, F_d$. The unique copula C of X (or F) with $C(u) = F(F_1^{\leftarrow}(u_1), \ldots, F_d^{\leftarrow}(u_d))$, is called an *elliptical copula*.

Example: Gaussian copulas are elliptical copulas
Let C_R^{Ga} be the copula of a d-dimensional normal distribution with correlation matrix R. Then $C_R^{Ga}(u) = \phi_R^d(\phi^{-1}(u_1), \ldots, \phi^{-1}(u_d))$ holds, where ϕ_R^d is the c.d.f. of a d-dimensional normal distribution with expected vector 0 and correlation matrix R, and ϕ^{-1} is the inverse of the standard normal distribution function.
Since the normal distribution is elliptic, the Gaussian copula C_R^{Ga} is by definition an elliptic copula.

Elliptical copulas (contd.)

Definition: Let $X \sim E_d(\mu, \Sigma, \psi)$ be an elliptically distributed random vector with c.d.f. F and marginal distributions $F_1, F_2, \ldots, F_d$. The unique copula C of X (or F) with $C(u) = F(F_1^{\leftarrow}(u_1), \ldots, F_d^{\leftarrow}(u_d))$, is called an *elliptical copula*.

Example: Gaussian copulas are elliptical copulas
Let C_R^{Ga} be the copula of a d-dimensional normal distribution with correlation matrix R. Then $C_R^{Ga}(u) = \phi_R^d(\phi^{-1}(u_1), \ldots, \phi^{-1}(u_d))$ holds, where ϕ_R^d is the c.d.f. of a d-dimensional normal distribution with expected vector 0 and correlation matrix R, and ϕ^{-1} is the inverse of the standard normal distribution function.
Since the normal distribution is elliptic, the Gaussian copula C_R^{Ga} is by definition an elliptic copula.

In the bivariate case we have:

$$C_R^{Ga}(u_1, u_2) = \int_{-\infty}^{\phi^{-1}(u_1)} \int_{-\infty}^{\phi^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \exp\left\{ \frac{-(x_1^2 - 2\rho x_1 x_2 + x_2^2)}{2(1-\rho^2)} \right\} dx_1 dx_2,$$

where $\rho \in (-1, 1)$.

Another example of elliptical copulas: the t-copula

Another example of elliptical copulas: the t-copula

Definition: Let $X \stackrel{d}{=} \mu + \frac{\sqrt{\alpha}}{\sqrt{S}} AZ \sim t_d(\alpha, \mu, \Sigma)$, where $\mu \in \mathbb{R}^d$, $\alpha \in \mathbb{N}$, $\alpha > 1$, $S \sim \chi^2_\alpha$, $A \in \mathbb{R}^{d \times k}$ with $AA^t = \Sigma$, $Z \sim N_k(0, I_k)$, and S and Z independent. We say that X has a d-dimensional t-distribution with expectation μ (for $\alpha > 1$) and covariance matrix $Cov(X) = \frac{\alpha}{\alpha - 2}\Sigma$. ($\alpha > 2$ should hold, $Cov(X)$ does not exist for $\alpha \leq 2$.)

Another example of elliptical copulas: the t-copula

Definition: Let $X \stackrel{d}{=} \mu + \frac{\sqrt{\alpha}}{\sqrt{S}} AZ \sim t_d(\alpha, \mu, \Sigma)$, where $\mu \in \mathbb{R}^d$, $\alpha \in \mathbb{N}$, $\alpha > 1$, $S \sim \chi^2_\alpha$, $A \in \mathbb{R}^{d \times k}$ with $AA^t = \Sigma$, $Z \sim N_k(0, I_k)$, and S and Z independent. We say that X has a d-dimensional t-distribution with expectation μ (for $\alpha > 1$) and covariance matrix $Cov(X) = \frac{\alpha}{\alpha-2}\Sigma$. ($\alpha > 2$ should hold, $Cov(X)$ does not exist for $\alpha \leq 2$.)

Definition: The (unique) copula $C^t_{\alpha,R}$ of X is called t-copula:

$$C^t_{\alpha,R}(u) = t^d_{\alpha,R}(t_\alpha^{-1}(u_1), \ldots, t_\alpha^{-1}(u_d)).$$

$R_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$, $i, j = 1, 2 \ldots, d$, is the correlation matrix of AZ.

$t^d_{\alpha,R}$ is the cdf of $\frac{\sqrt{\alpha}}{\sqrt{S}} Y$, where $S \sim \chi^2_\alpha$, $Z \sim N_k(0, R)$, and S, Y are independent. t_α are the marginal distributions of $t^d_{\alpha,R}$.

Another example of elliptical copulas: the t-copula

Definition: Let $X \stackrel{d}{=} \mu + \frac{\sqrt{\alpha}}{\sqrt{S}} AZ \sim t_d(\alpha, \mu, \Sigma)$, where $\mu \in \mathbb{R}^d$, $\alpha \in \mathbb{N}$, $\alpha > 1$, $S \sim \chi^2_\alpha$, $A \in \mathbb{R}^{d\times k}$ with $AA^t = \Sigma$, $Z \sim N_k(0, I_k)$, and S and Z independent. We say that X has a d-dimensional t-distribution with expectation μ (for $\alpha > 1$) and covariance matrix $Cov(X) = \frac{\alpha}{\alpha-2}\Sigma$. ($\alpha > 2$ should hold, $Cov(X)$ does not exist for $\alpha \leq 2$.)

Definition: The (unique) copula $C^t_{\alpha,R}$ of X is called t-copula:

$$C^t_{\alpha,R}(u) = t^d_{\alpha,R}(t^{-1}_\alpha(u_1), \ldots, t^{-1}_\alpha(u_d)).$$

$R_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$, $i, j = 1, 2 \ldots, d$, is the correlation matrix of AZ.

$t^d_{\alpha,R}$ is the cdf of $\frac{\sqrt{\alpha}}{\sqrt{S}} Y$, where $S \sim \chi^2_\alpha$, $Z \sim N_k(0, R)$, and S, Y are independent. t_α are the marginal distributions of $t^d_{\alpha,R}$.

In the bivariate case ($d = 2$):

$$C^t_{\alpha,R}(u_1, u_2) = \int_{-\infty}^{t_\alpha^{-1}(u_1)} \int_{-\infty}^{t_\alpha^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \left\{1 + \frac{x_1^2 - 2\rho x_1 x_2 + x_2^2}{\alpha(1-\rho^2)}\right\}^{-\frac{\alpha+2}{2}} dx_1 dx_2$$

for $\rho \in (-1, 1)$. R_{12} is the linear correlation coefficient of the corresponding bivariate t_α-distribution for $\alpha > 2$.

Further properties of copulas

Further properties of copulas

Definition: (Radial symmetry)
A d-dimensional random vector X (or a d-variate distribution function) is called *radial symmetric around a*, for some $a \in \mathbb{R}^d$, iff $X - a \stackrel{d}{=} a - X$.

Further properties of copulas

Definition: (Radial symmetry)
A d-dimensional random vector X (or a d-variate distribution function) is called *radial symmetric around* a, for some $a \in \mathbb{R}^d$, iff $X - a \stackrel{d}{=} a - X$.

Example: An elliptically distributed random vector $X \sim E_d(\mu, \Sigma, \psi) \in \mathbb{R}^d$ is radial symmetric around μ.

Further properties of copulas

Definition: (Radial symmetry)
A d-dimensional random vector X (or a d-variate distribution function) is called *radial symmetric around a*, for some $a \in \mathbb{R}^d$, iff $X - a \stackrel{d}{=} a - X$.

Example: An elliptically distributed random vector $X \sim E_d(\mu, \Sigma, \psi) \in \mathbb{R}^d$ is radial symmetric around μ.

Definition: (Radial symmetry of copulas)
A copula C is called radial symmetric iff

$$(U_1 - 0.5, \ldots, U_d - 0.5) \stackrel{d}{=} (0.5 - U_1, \ldots, 0.5 - U_d) \Longleftrightarrow U \stackrel{d}{=} 1 - U,$$

where $(U_1, U_2, \ldots, U_d)$ is a random vector with distribution function C.
For a radial symmetric copula $C = \hat{C}$ holds.

Further properties of copulas

Definition: (Radial symmetry)
A d-dimensional random vector X (or a d-variate distribution function) is called *radial symmetric around a*, for some $a \in \mathbb{R}^d$, iff $X - a \stackrel{d}{=} a - X$.

Example: An elliptically distributed random vector $X \sim E_d(\mu, \Sigma, \psi) \in \mathbb{R}^d$ is radial symmetric around μ.

Definition: (Radial symmetry of copulas)
A copula C is called radial symmetric iff

$$(U_1 - 0.5, \ldots, U_d - 0.5) \stackrel{d}{=} (0.5 - U_1, \ldots, 0.5 - U_d) \Longleftrightarrow U \stackrel{d}{=} 1 - U,$$

where $(U_1, U_2, \ldots, U_d)$ is a random vector with distribution function C.
For a radial symmetric copula $C = \hat{C}$ holds.

Example: Elliptical copulas are radial symmetric.

Further properties of copulas

Definition: (Radial symmetry)
A d-dimensional random vector X (or a d-variate distribution function) is called *radial symmetric around a*, for some $a \in \mathbb{R}^d$, iff $X - a \stackrel{d}{=} a - X$.

Example: An elliptically distributed random vector $X \sim E_d(\mu, \Sigma, \psi) \in \mathbb{R}^d$ is radial symmetric around μ.

Definition: (Radial symmetry of copulas)
A copula C is called radial symmetric iff

$$(U_1 - 0.5, \ldots, U_d - 0.5) \stackrel{d}{=} (0.5 - U_1, \ldots, 0.5 - U_d) \Longleftrightarrow U \stackrel{d}{=} 1 - U,$$

where $(U_1, U_2, \ldots, U_d)$ is a random vector with distribution function C.
For a radial symmetric copula $C = \hat{C}$ holds.

Example: Elliptical copulas are radial symmetric.

The Gumbel and Clayton Copulas are not radial symmetric. Why?

The density function of a copula

The density function of a copula

Not every copula has a density function. For example the co-monotony copula M and the anti-monotony copula W do not have a density function.

The density function of a copula

Not every copula has a density function. For example the co-monotony copula M and the anti-monotony copula W do not have a density function.

If the density function c of a copula C exists, then we have

$$c(u_1, u_2, \ldots, u_d) = \frac{\partial C(u_1, u_2, \ldots, u_d)}{\partial u_1 \partial u_2 \ldots \partial u_d}.$$

The density function of a copula

Not every copula has a density function. For example the co-monotony copula M and the anti-monotony copula W do not have a density function.

If the density function c of a copula C exists, then we have

$$c(u_1, u_2, \ldots, u_d) = \frac{\partial C(u_1, u_2, \ldots, u_d)}{\partial u_1 \partial u_2 \ldots \partial u_d}.$$

Let C be the copula of a distribution F with marginal distributions F_1,...,F_d. By differentiating

$$C(u_1, \ldots, u_d) = F(F_1^{\leftarrow}(u_1), \ldots, F_d^{\leftarrow}(u_d))$$

we obtain the density c of C:

$$c(u_1, \ldots, u_d) = \frac{f(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))}{f_1(F_1^{-1}(u_1)) \ldots f_d(F_d^{-1}(u_d))}$$

where f is the density function of F, f_i are the marginal density functions, and F_i^{-1} are the inverse functions of F_i, for $1 \leq i \leq d$,

Exchangeability

Exchangeability

Definition:
A random vector X is called *'exchangeable* iff $(X_1, \ldots, X_d) \stackrel{d}{=} (X_{\pi(1)}, \ldots, X_{\pi(d)})$ for any permutation $(\pi(1), \pi(2), \ldots, \pi(d))$ of $(1, 2, \ldots, d)$.
A copula C is called *'exchangeable* iff C is the distribution function of an exchangeable random vector (with uniform marginal distributions on $[0, 1]$).

Exchangeability

Definition:

A random vector X is called *'exchangeable* iff $(X_1, \ldots, X_d) \stackrel{d}{=} (X_{\pi(1)}, \ldots, X_{\pi(d)})$ for any permutation $(\pi(1), \pi(2), \ldots, \pi(d))$ of $(1, 2, \ldots, d)$.

A copula C is called *'exchangeable* iff C is the distribution function of an exchangeable random vector (with uniform marginal distributions on $[0, 1]$).

For such a copula $C(u_1, u_2, \ldots, u_d) = C(u_{\pi(1)}, u_{\pi(2)}, \ldots, u_{\pi(d)})$ holds for any permutation $(\pi(1), \pi(2), \ldots, \pi(d))$ of $(1, 2, \ldots, d)$.

Exchangeability

Definition:

A random vector X is called *'exchangeable* iff
$(X_1, \ldots, X_d) \stackrel{d}{=} (X_{\pi(1)}, \ldots, X_{\pi(d)})$ for any permutation $(\pi(1), \pi(2), \ldots, \pi(d))$ of $(1, 2, \ldots, d)$.
A copula C is called *'exchangeable* iff C is the distribution function of an exchangeable random vector (with uniform marginal distributions on $[0, 1]$).

For such a copula $C(u_1, u_2, \ldots, u_d) = C(u_{\pi(1)}, u_{\pi(2)}, \ldots, u_{\pi(d)})$ holds for any permutation $(\pi(1), \pi(2), \ldots, \pi(d))$ of $(1, 2, \ldots, d)$.

Examples of exchangeable copulas:

Gumbel, Clayton, and also the Gaussian copula C_P^{Ga} and the t-Copula $C_{\nu,P}^t$, if P is an *equicorrelation matrix*, i.e. $R = \rho J_d + (1 - \rho) I_d$.
$J_d \in \mathbb{R}^{d \times d}$ is a matrix consisting only of ones, and $I_d \in \mathbb{R}^{d \times d}$ is the d-dimensional identity matrix.

Exchangeability

Definition:

A random vector X is called *'exchangeable* iff $(X_1, \dots, X_d) \stackrel{d}{=} (X_{\pi(1)}, \dots, X_{\pi(d)})$ for any permutation $(\pi(1), \pi(2), \dots, \pi(d))$ of $(1, 2, \dots, d)$.

A copula C is called *'exchangeable* iff C is the distribution function of an exchangeable random vector (with uniform marginal distributions on $[0, 1]$).

For such a copula $C(u_1, u_2, \dots, u_d) = C(u_{\pi(1)}, u_{\pi(2)}, \dots, u_{\pi(d)})$ holds for any permutation $(\pi(1), \pi(2), \dots, \pi(d))$ of $(1, 2, \dots, d)$.

Examples of exchangeable copulas:

Gumbel, Clayton, and also the Gaussian copula C_P^{Ga} and the t-Copula $C_{\nu,P}^t$, if P is an *equicorrelation matrix*, i.e. $R = \rho J_d + (1-\rho) I_d$.
$J_d \in \mathbb{R}^{d \times d}$ is a matrix consisting only of ones, and $I_d \in \mathbb{R}^{d \times d}$ is the d-dimensional identity matrix.

For bivariate exchangeable copulas we have:

$$P(U_2 \leq u_2 | U_1 = u_1) = P(U_1 \leq u_2 | U_2 = u_1).$$

Tail dependence coefficients of elliptical copulas

Tail dependence coefficients of elliptical copulas

Theorem: Let $(X_1, X_2)^T$ be a normally distributed random vector. Then $\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 0$ holds.

Tail dependence coefficients of elliptical copulas

Theorem: Let $(X_1, X_2)^T$ be a normally distributed random vector. Then $\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 0$ holds.

Corollary: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and let C_ρ^{Ga} be a Gaussian copula, where ρ is the linear correlation coefficient of X_1 and X_2. The $\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 0$ holds.

Tail dependence coefficients of elliptical copulas

Theorem: Let $(X_1, X_2)^T$ be a normally distributed random vector. Then $\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 0$ holds.

Corollary: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and let C_ρ^{Ga} be a Gaussian copula, where ρ is the linear correlation coefficient of X_1 and X_2. The $\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 0$ holds.

Theorem: Let $(X_1, X_2)^T \sim t_2(0, \nu, R)$ be a random vector with a t-distribution and ν degrees of freedom, expectation 0 and linear correlation matrix R. For $R_{12} > -1$ we have

$$\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-R_{12}}}{\sqrt{1+R_{12}}}\right)$$

Tail dependence coefficients of elliptical copulas

Theorem: Let $(X_1, X_2)^T$ be a normally distributed random vector. Then $\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 0$ holds.

Corollary: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and let C_ρ^{Ga} be a Gaussian copula, where ρ is the linear correlation coefficient of X_1 and X_2. The $\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 0$ holds.

Theorem: Let $(X_1, X_2)^T \sim t_2(0, \nu, R)$ be a random vector with a t-distribution and ν degrees of freedom, expectation 0 and linear correlation matrix R. For $R_{12} > -1$ we have

$$\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-R_{12}}}{\sqrt{1+R_{12}}}\right)$$

The proof is similar to the proof of the analogous theorem about the Gaussian copulas.
Hint:

$$X_2|X_1 = x \sim \left(\frac{\nu+1}{\nu+x^2}\right)^{1/2}\frac{X_2-\rho x}{\sqrt{1-\rho^2}} \sim t_{\nu+1}$$

.

Tail dependence (contd.) and rank correlation of elliptical copulas

Tail dependence (contd.) and rank correlation of elliptical copulas

Corollary: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and a t-copula $C^t_{\nu,R}$ with ν degrees of freedom and and correlation matrix R. Then we have

$$\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-R_{12}}}{\sqrt{1+R_{12}}}\right).$$

Tail dependence (contd.) and rank correlation of elliptical copulas

Corollary: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and a t-copula $C^t_{\nu,R}$ with ν degrees of freedom and and correlation matrix R. Then we have $\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-R_{12}}}{\sqrt{1+R_{12}}}\right)$.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and a Gaussian copula C^{Ga}_ρ, where ρ is the linear correlation coefficient of X_1 and X_2. Then we have $\rho_\tau(X_1, X_2) = \frac{2}{\pi}\arcsin\rho$ und $\rho_S(X_1, X_2) = \frac{6}{\pi}\arcsin\frac{\rho}{2}$.

Tail dependence (contd.) and rank correlation of elliptical copulas

Corollary: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and a t-copula $C^t_{\nu,R}$ with ν degrees of freedom and and correlation matrix R. Then we have
$\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-R_{12}}}{\sqrt{1+R_{12}}}\right)$.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and a Gaussian copula C^{Ga}_ρ, where ρ is the linear correlation coefficient of X_1 and X_2. Then we have $\rho_\tau(X_1, X_2) = \frac{2}{\pi}\arcsin\rho$ und $\rho_S(X_1, X_2) = \frac{6}{\pi}\arcsin\frac{\rho}{2}$.

Theorem: Let $X \sim E_d(\mu, \Sigma, \psi)$ be an elliptically distributed random vector with continuous marginal distributions. Then the following holds $\rho_\tau(X_i, X_j) = \frac{2}{\pi}\arcsin R_{ij}$, with $R_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$ for $i, j = 1, 2, \ldots, d$.

Tail dependence (contd.) and rank correlation of elliptical copulas

Corollary: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and a t-copula $C^t_{\nu,R}$ with ν degrees of freedom and and correlation matrix R. Then we have
$\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-R_{12}}}{\sqrt{1+R_{12}}}\right)$.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and a Gaussian copula C^{Ga}_ρ, where ρ is the linear correlation coefficient of X_1 and X_2. Then we have $\rho_\tau(X_1, X_2) = \frac{2}{\pi}\arcsin\rho$ und $\rho_S(X_1, X_2) = \frac{6}{\pi}\arcsin\frac{\rho}{2}$.

Theorem: Let $X \sim E_d(\mu, \Sigma, \psi)$ be an elliptically distributed random vector with continuous marginal distributions. Then the following holds $\rho_\tau(X_i, X_j) = \frac{2}{\pi}\arcsin R_{ij}$, with $R_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$ for $i, j = 1, 2, \ldots, d$.

Corollary: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and an elliptical copula $C^E_{\mu,\Sigma,\psi}$. Then we have
$\rho_\tau(X_1, X_2) = \frac{2}{\pi}\arcsin R_{12}$, with $R_{12} = \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}}$.

Tail dependence (contd.) and rank correlation of elliptical copulas

Corollary: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and a t-copula $C^t_{\nu,R}$ with ν degrees of freedom and and correlation matrix R. Then we have
$\lambda_U(X_1, X_2) = \lambda_L(X_1, X_2) = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\frac{\sqrt{1-R_{12}}}{\sqrt{1+R_{12}}}\right)$.

Theorem: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and a Gaussian copula C^{Ga}_ρ, where ρ is the linear correlation coefficient of X_1 and X_2. Then we have $\rho_\tau(X_1, X_2) = \frac{2}{\pi}\arcsin\rho$ und $\rho_S(X_1, X_2) = \frac{6}{\pi}\arcsin\frac{\rho}{2}$.

Theorem: Let $X \sim E_d(\mu, \Sigma, \psi)$ be an elliptically distributed random vector with continuous marginal distributions. Then the following holds $\rho_\tau(X_i, X_j) = \frac{2}{\pi}\arcsin R_{ij}$, with $R_{ij} = \frac{\Sigma_{ij}}{\sqrt{\Sigma_{ii}\Sigma_{jj}}}$ for $i, j = 1, 2, \ldots, d$.

Corollary: Let $(X_1, X_2)^T$ be a random vector with continuous marginal distributions and an elliptical copula $C^E_{\mu,\Sigma,\psi}$. Then we have
$\rho_\tau(X_1, X_2) = \frac{2}{\pi}\arcsin R_{12}$, with $R_{12} = \frac{\Sigma_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}}$.

See McNeil et al. (2005) for a proof of the three last results.