Risk theory and risk management in actuarial science
Winter term 2016/17
6th work sheet

31. **Equivalent threshold models**
Let $X = (X_1, X_2, \ldots, X_m)'$ be an m-dimensional random vector and let $D \in \mathbb{R}^{m \times n}$ be a deterministic matrix with elements d_{ij} such that for every i, $1 \le i \le m$, the elements of the i-th row form a set of increasing thresholds satisfying $d_{i,1} < d_{i2} \ldots < d_{in}$. Introduce additionally $d_{i0} = -\infty$, $d_{i,n+1} = +\infty$ and set
$$S_i = j \iff d_{ij} < X_i \le d_{i,j+1}\,, \text{for } j \in \{0, \ldots, n\}, i \in \{1, \ldots, m\}.$$
Then (X, D) is a said to define a *threshold model for the state vector* $S = (S_1, \ldots, S_m)'$. We refere to X as the *vector of critical variables* and denote its marginal distribution functions by $F_i(x) = P(X_i \le x)$, for $i \in \{1, 2, \ldots, m\}$. The i-th row of D contains the critical thresholds for firm i. By definition, default (corresponding to event $S_i = 0$) occurs iff $X_i \le d_{i1}$, thus the default probability of company i is given by $\bar{p}_i := F_i(d_{i1})$. We denote by $\rho(Y_i, Y_j)$ the *default correlation* of two firms $i \neq j$; this quantity depends on $E(Y_i, Y_j)$ (how?) which in turn depends on the joint distribution of (X_i, X_j), and hence on the copula of $(X_1, X_2, \ldots, X_m)'$. (Notice that in general the latter is not fully determined by the *asset correlation* $\rho(X_i, X_j)$.)
Two threshold models (X, D) and $(\tilde{X}, \tilde{D})$ for the state vectors S and $\tilde{S}$, respectively, are called equivalent, iff S and $\tilde{S}$ have the same probability distribution.
Show that two threshod models (X, D) and $(\tilde{X}, \tilde{D})$ with state vectors S and $\tilde{S}$, respectively, are equivalent if the following conditions hold:
    (a) The marginal distributions of the random vectors S and $\tilde{S}$ coincide, i.e. $P(S_i = j) = P(\tilde{S}_i = j)$, for all $j \in \{1, \ldots, n\}$, $i \in \{1, \ldots, m\}$.
    (b) X and $\tilde{X}$ admit the same copula C.

32. **Threshold models as mixture models**
We say that X *has a p-dimensional conditional independence structure with conditioning variable* Ψ if there is some $p < m$ and p-dimensional random vector $\Psi = (\Psi_1, \ldots, \Psi_m)'$ such that the random variables $X_1,\ldots, X_m$ are independent conditional on Ψ. Let (X, D) be a threshold model for the state vector S (cf. Exercise 31).
    (a) Assume that X has a p-dimensional conditional independence structure with conditioning variable Ψ. Show that then the default indicators $Y_i = I_{\{X_i \le d_{i1}\}}$ follow a Bernoulli mixture model with factors Ψ, where the conditional default probailities are given by $p_i(\psi) = P(X_i \le d_{i1} | \Psi = \psi)$.
    (b) Consider the following model for the distribution of the critical variables X. Start with an m-dimensional multivariate normal vector $Z \sim N_m(0, \Sigma)$ and a positive scalar random variable W, which is independent on Z. Assume the following structure of X: $X = m(W) + \sqrt{W} Z$, where $m\colon [0, \infty) \to \mathbb{R}^m$ is a measurable function. (Recall that in the case that $m(W)$ takes a constant value μ not depending on W we get the normal variance mixture distribution for X. If, moreover, W has an inverse gamma distribution $W \sim Ig(\frac{\nu}{2}, \frac{\nu}{2})$, or equivalently, $\frac{\nu}{W} \sim \chi^2_\nu$, than we get the multivariate t distribution.) Suppose now that Z follows the linear factor model $Z = BF + \epsilon$ for a factor $F \sim N_p(0, \Omega)$, a loading matrix $B \in \mathbb{R}^{m \times p}$, and independently distributed random variables $\epsilon_1,\ldots,\epsilon_m$, which are also independent on F. Show that X has a $(p+1)$-dimensional conditional independent structure with $\Psi = (F_1, \ldots, F_p, W)$ and conditional default probabilities
$$p_i(\psi) = P(X_i \le d_{i1} | \Psi = \psi) = \Phi\Big(\frac{d_{i1} - m_i(w) - \sqrt{w} b_i' f}{\sqrt{w \upsilon_i}}\Big)\,, \qquad (1)$$
where $m_i(w)$ is the i-th component of $m(w)$, b_i is the i-th row of B and υ_i is the i-th diagonal element of the diagonal covariance matrix of $(\epsilon_1, \ldots, \epsilon_m)'$.

(c) Consider the special case of the Moody's KMV model where $X = Z$ and $\Psi = F$. Standartize the critical variables X_i to have variance one and reparametrize the formula (1) in terms of the individual default probabilities $\bar{p}_i$ and the systematic variance component $\beta_i = b_i'\Omega b_i = 1 - \upsilon_i$ so as to obtain

$$p_i(\psi) = \Phi\Big(\frac{\phi^{-1}(\bar{p}_i) - b_i'\psi}{\sqrt{1-\beta_i}}\Big).$$

Observe that the individual default probabilities $p_i(\psi)$ have a probit normal distribution with parameters $\mu_i = \Phi^{-1}(\bar{p}_i)/\sqrt{1-\beta_i}$ and $\sigma_i = \sqrt{\beta_i/(1-\beta_i)}$.

33. (Model risk: the impact of the choice of the copula)
We compare two threshold models with different copulas of the critical variables, the Gauss copula and the t copula. Consider a standard normal random variable F and an iid sequence $\epsilon_1, \ldots, \epsilon_m$ of standard normal variables independent on F, as well as an asset correlation parameter $\rho \in [0,1]$. Define a random vector $Z_i = \sqrt{\rho}F + \sqrt{1-\rho}\epsilon_i$. Consider further a random variable $W \sim Ig(\frac{\nu}{2}, \frac{\nu}{2})$ independent on Z. In the case of the t copula the critical variables are given as $X_i = \sqrt{W}Z_i$ and in the case of the Gauss copula they are given as $X_i = Z_i$, for $i \in \{1, \ldots, m\}$. In both cases the thresholds are chosen such that $P(Y_i = 1) = \pi$ for all $i \in \{1, \ldots, m\}$ and some $\pi \in (0,1)$. Note that the correlation matrix P of X is identical in both models; it is an equicorrelation matrix with all off-diagonal elements equal to ρ.

Consider three groups of credits of decreasing quality named A, B and C, distinguished in terms of the individual default probability and the asset correlation coefficient: $\pi = 0.06\%$ and $\rho = 2.58\%$ in A, $\pi = 0.50\%$ and $\rho = 3.80\%$ in B, and $\pi = 7.50\%$ and $\rho = 9.21\%$ in C. Consider portfolios of size $m = 10000$ for each group and measure the risk of each portfolio by assessing two quantiles of the number of defaults $M := \sum_{i=1}^m Y_i$ at the level 95% and 99%, respectively. The quantiles should be estimated by simulating the Bernoulli mixed models corresponding to the threshold models (cf. Exercise 32). Compare the results obtained for three different values of the degrees of freedom parameter ν for each group: $\nu = 10$, $\nu = 50$ and $\nu = \infty$ (the latter value corresponds to the Gauss copula). Compile a three by three table will all computed quantiles.

34. (Exponential tilting for the normal distribution)
Apply the exponential tilting approach to estimate the tail probability $P(X > c)$ for a standard normally distributed random variable $X \sim N(0,1)$ and $c >> 0$. Determine the parameter t of the tilted distribution density g_t in this case (cf. lecture).