Operations Research
winter term 2017/2018

5th work sheet (multicriteria optimization)

29. (Ehrgott et al., 1997[1])
Let $X \subseteq \mathrm{I\!R}^n$, $f: X \to \mathrm{I\!R}$, and $\bar{x} \in X$. The set $L_{\leq}(f(\bar{x})) = \{x \in X: f(x) \leq f(\bar{x})\}$ is called a *level set* of f in $\bar{x}$, the set $L_{=}(f(\bar{x})) = \{x \in X: f(x) = f(\bar{x})\}$ is called *level curve* of f in $\bar{x}$ and $L_{<}(f(\bar{x})) = \{x \in X: f(x) < f(\bar{x})\}$ is called *strict level set* of f in $\bar{x}$. Consider an MCOP $(X, f, \mathrm{I\!R}^Q)/\mathrm{id}/(\mathrm{I\!R}^Q, <)$. Let $x^* \in X$, $f = (f_1, f_2, \ldots, f_q)^T$ and $y_q := f_q(x^*)$ for $q = 1, 2, \ldots, Q$. Show that

   (a) x^* is strict Pareto optimal if and only if $\cap_{q=1}^{Q} L_{\leq}(y_q) = \{x^*\}$, where $L_{\leq}(y_q)$ is the level set of f_q in x^*, $q = 1, 2, \ldots, Q$.

   (b) x^* is Pareto optimal if and only if $\cap_{q=1}^{Q} L_{\leq}(y_q) = \cap_{q=1}^{Q} L_{=}(y_q)$, where $L_{\leq}(y_q)$ is as in (a) and $L_{=}(y_q)$ is the level curve of f_q in y_q, $q = 1, 2, \ldots, Q$.

   (c) x^* ist weakly Pareto optimal if and only if $\cap_{q=1}^{Q} L_{<}(y_q) = \emptyset$, where $L_{<}(y_q)$ is the strict level set of f_q in y_q, $q = 1, 2, \ldots, Q$.

30. Provide an example in which the following relationships hold, respectively:

   (a) $S(Y) \subset Y_{eff} \subset S_0(Y)$ where both inclusions are strict and $S(Y)$, $S_0(Y)$ are defined as in the lecture.

   (b) $S(Y) \cup S_0'(Y) = Y_{eff} = S_0(Y)$, where

$$S_0'(Y) = \left\{ y' \in Y: \exists \lambda \in \mathrm{I\!R}^Q_+ \setminus \{0\} \text{ such that } \{y'\} = S(\lambda, Y) \right\}.$$

   $S(\lambda, Y)$, $S(Y)$ and $S_o(Y)$ are defined as in the lecture, namely

$$S(\lambda, Y) = \operatorname{argmin}\{\langle \lambda, y\rangle : y \in Y\}, \; S(Y) = \cup_{\lambda \in Int(\mathrm{I\!R}^Q_+)} S(\lambda, Y), \; \text{ and } S_0(Y) = \cup_{\lambda \in \mathrm{I\!R}^Q_+ \setminus \{0\}} S(\lambda, Y).$$

31. Let $X = \{(x_1, x_2) \in \mathrm{I\!R}^2: x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 100, 2x_1 + x_2 \leq 150\}$, $f_1(x_1, x_2) = -6x_1 - 4x_2$ and $f_2(x_1, x_2) = -x_1$. Solve the *ϵ-constrained problem* $P_1(\epsilon)$ for $\epsilon = 0$ cf. lecture for the definition of ϵ-constrained problems). Use the method of Benson to check whether the optimal solution x^* of $P_1(0)$ is Pareto-optimal to $(X, f, \mathrm{I\!R}^2)/\mathrm{id}/(\mathrm{I\!R}^2, <)$ or not.

32. Consider $X = \{(x_1, x_2) \in \mathrm{I\!R}^2: (x_1 - 1)^2 + (x_2 - 1)^2 \leq 1\}$ and $f: X \to \mathrm{I\!R}^2$ with $f(x) = x$, $\forall x \in X$. Solve the problem $P_2(\epsilon, \mu)$ (cf. lecture) with $\epsilon = 0$ in order to illustrate that there exist (non-proper) Pareto optimal solutions to $(X, f, \mathrm{I\!R}^2)/\mathrm{id}/(\mathrm{I\!R}^2, \leq)$ which can not be obtained as optimal solutions of $P_2(\epsilon, \mu)$ for some $0 \leq \mu < \infty$. Could this Pareto optimal solution of $(X, f, \mathrm{I\!R}^2)/\mathrm{id}/(\mathrm{I\!R}^2, \leq)$ be obtained as an optimal solution of $P_2(\epsilon, \mu)$ with $0 \leq \mu < \infty$ for some other value of $\epsilon \neq 0$?

33. Consider $(X, f, \mathrm{I\!R}^n)/\mathrm{id}/(\mathrm{I\!R}^Q, \leq)$ and assume $0 < \min_{x \in X} f_k(x)$, for all $k = 1, 2, \ldots, Q$. Prove that $x \in X_{\mathrm{wPar}}$ if and only if x is an optimal solution of $\min_{x \in X} \max_{k=1,2,\ldots,Q} \lambda_k f_k(x)$, for some $\lambda \in \mathrm{int}(\mathrm{I\!R}^Q_+)$.

34. Consider finding a compromise solution by maximizing the distance to the nadir point. Let $||\cdot||$ be a norm. Show that an optimal solution of the problem $\max\{||f(x) - y^N||: x \in X\}$ is weakly Pareto optimal to $(X, f, \mathrm{I\!R}^n)/\mathrm{id}/(\mathrm{I\!R}^Q, \leq)$. Give a condition under which an optimal solution of the above maximization problem is guaranted to be a Pareto optimal solution of $(X, f, \mathrm{I\!R}^n)/\mathrm{id}/(\mathrm{I\!R}^Q, \leq)$.

[1] M. Ehrgott, Multicriteria Optimization, Second Edition, Springer, Berlin-Heidelberg-New York, 2005

35. Solve the problem in example 31 by means of the compromise programming approach. Use $\lambda = (1/2, 1/2)$ and identify the solution of CP_p^w für $p = 1, 2, \infty$.

36. Let $Y = \{y = (y_1, y_2) \in \mathbb{R}^2_+ \colon \mathrm{y}_1^2 + \mathrm{y}_2^2 \geq 1\}$. Show the existence of a parameter p, $1 < p < \infty$, such that the following equality holds:

$$Y_{\text{eff}} = \cup_{w \in \Lambda^0} A(\lambda, p, Y).$$

Use the ideal point y^I or the utopic point y^U in the definition of $A(w, p, Y)$ and N_p^λ.