Operations Research
winter term 2019/2020

5th work sheet (dynamic programming)

20. Let the demand r_i on a certain component be given as follows for the next five periods i, $1 \leq i \leq 5$: $r_1 = 2$, $r_2 = 4$, $r_3 = 2$, $r_4 = 2$ und $r_5 = 4$. The components have to be produced and the production involves setup costs of 5 Euro, production costs of 1 Euro per piece, and inventory costs of $0,30$ Euro per piece and month. Assume that the whole production of a month happens at the beginning of the month and that in every month the whole demand is delivered to the customer immediately after the production at the beginning of the month. Determine by means of dynamic programming a production plan which fulfills the demand and minimizes the total costs.

21. A taxi company consumes 8500 liter fuel per month. The consumption happens continuously at a constant rate. The costs of the fuel amount to 1.20 Euro per liter, the fixed ordering costs and the inventory costs amount to 1000 Euro per order and 1 Cent per liter and month, respectively. Use the EOQ model (cf. the lecture) to determine an ordering plan (i.e. when to order and which quantity to order, respectively) which minimizes the overall costs and avoids shortfalls. How would you obtain an optimal solution if the optimal cycle length is required to be an integer number (of months)?

22. Three teams of scientists are dealing with a difficult, yet unsolved, problem. The failure probability of the teams, denoted by i, $1 \leq i \leq 3$, is estimated to be $0,4$, $0,6$ and $0,8$, respectively. The efforts to solve the problem will be intensified and two more scientists will deal with the problem as members of some of the three teams. Table 22 shows for each (extended) team the estimated probability of its failure depending on the additonal number of scientists it gets assigned. Determine the best assignment of the two additional scientists to teams such that the estimated probability that all three teams fail is minimized. Solve this problem by means of dynamic programming. Assume that the failures of the teams happen independently and notice that a Bellman-like equation can also be written in the case where the overall objective function is not the sum but the product of the costs in the single periods.

Number of additional scientists assigned	Estimated failure probability		
	Team		
	1	2	3
0	0.4	0.6	0.8
1	0.2	0.4	0.5
2	0.15	0.2	0.3

Table 1: Data for Problem 22

23. The workload in a local company depends heavily on strong saisonal fluctuations. It is not desirable to dismiss a part of the staff in the periods with lower workload, and it is also not desirable to pay the wages of the peak-period all over the year, if this is not indispensable. Moreover, the management is principally against regular overtime hours. Since the production has to be done on demand it is not possible to build some inventory in the more quiet periods. In these circumstances it is difficult to determine an optimal employment policy.

    The estimated number of employees needed in the four seasons of the coming year is given as follows

Saison	Spring	Summer	Automn	Winter	Spring
Employees needed	255	270	240	200	255

and the number of employees should in no way decrease under the above given levels. The extra costs for employing more people than (presumably) needed are estimated to be 2000 Euro per employee and season. Further it is assumed that the recruitement and dismission costs are given by the squared difference between the numbers of employees in the two consecutive seasons multiplied by 130. Notice that it is possible to employ part time staff; in this case the corresponding costs would be proportional to the emplyoment hours.

The management wants to determine the levels of emplyement for every season of the following year such that the extra costs are minimized. Solve this problem by means of dynamic programming.

24. Formulate the following optimization problem as a deterministic dynamic programming problem with a continuous state space and a finite time horizon (DDOCPF) and solve it by applying the value iteration algorithm (cf. lecture).

$$\begin{aligned} \max \quad & 3x_1 + 7x_2 + 6f(x_3) \\ \text{s.t.} \quad & \\ & x_1 + 3x_2 + 2x_3 \leq 6 \\ & x_1 + x_2 \leq 5 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

where $f\colon [0, +\infty) \to \mathbb{R}$ is given as follows:

$$f(x) = \begin{cases} 0 & x = 0 \\ -1 + x_3 & x > 0 \end{cases}$$

25. Formulate the following optimization problem as a deterministic dynamic optimization problem with a finite time horizon (DDOPF) and solve it by applying the value iteration algorithm (cf. lecture).

$$\begin{aligned} \max \quad & z = x_1 x_2^2 x_3^3 \\ \text{s.t.} \quad & \\ & x_1 + 2x_2 + 3x_3 \leq 10 \\ & x_1 \geq 1,\ x_2 \geq 1,\ x_3 \geq 1 \\ & x_1,\ x_2,\ x_3 \text{ integers} \end{aligned}$$

26. A company produces a delicate product by using a sofisticated technology which is not fully developed yet. Every piece of the product works well with probability 1/2 and is irreparably defective with probability 1/2, where the failures of the pieces are independent of each other. The company produces many pieces of the product in each production run and hopes that at least one working piece will be produced. The remaining pieces are worthless and will be discarded no matter whether they are defective or not. The production costs amount to 100 Euro per piece and the fixed costs amount to 300 Euro per production run. The time available until delivery allows to complete at most three production runs. The company has to pay a fine of 1600 Euro if it cannot deliver a working piece of the product on due time. Determine a production policy, i.e. the number of the production runs and the number of pieces produced in each production run, such that the expected overall cost of the company is minimized.

27. A popular game in Las Vegas is the one called "all or nothing": in every run of the game the player places a number of chips and either wins all of them or loses all of them. Assume that the probability to win a run of the the game is 2/3 and different runs of the game are independent. Consider a player who possesses three chips, plays at most three runs, and considers the possession of five chips at the end of the game as a victory. Determine a playing strategy, which maximizes the probability of a victory (according to the players own definition of a victory). A playing strategy specifies the number of the placed chips in every run of the game depending on the outcome of the previous runs.

28. Consider a single-product warehouse with a storage capacity of M, $M \in \mathbb{N}$ (i.e. the warehouse can store at most M product units at a time). The warehouse gets inspected at discrete points n in time,

$n = 0, 1, \ldots, N-1$, and at each inspection it will be decided whether to order a certain amount $b_n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ of the product and increase the stock or not. Assume that the ordered amount of the product reaches the warehouse immediately, i.e. at the very same moment in which the order has been placed. The demand on the product in the time interval $[n, n+1)$ is the realisation of a random variable Y_n, $n = 0, 1, \ldots, N-1$. The random variables Y_n, $n = 0, 1, \ldots, N-1$, are assumed to be identically and independently distributed with a density given as $P(Y_n = x) = q(x)$, $\forall x \in \mathbb{N}_0$ and $n = 0, 1, \ldots, N-1$. Moreover it is assumed that these random variables have a finite expected value. Assume further that the demand is fulfilled immediately after the arrival of the ordered products in the warehouse. If there is not enough stock to fulfill the demand in that moment, then the missed demand will be considered again, right after the arrival of the next order. The odering costs amount to cb_n for every placed order b_n, $n = 0, 1, \ldots, N-1$, where c is a given constant. There are also storage costs and shortfall costs given as

$$l(z_n) = \begin{cases} l_1 \cdot z & z \geq 0 \text{ (stock)} \\ -l_2 \cdot z & z < 0 \text{ (shortfall)} \end{cases},$$

where $l_1, l_2 \geq 0$ are prespecified constants, and z_n is the stock or shortfall immediately after the arrival of the order and the delivery of the demand at the beginning of the time interval $[n, n+1)$. The goal is to determine the minimum of the overall expected costs and an optimal ordering policy. Formulate this problem as a stochastic dynamic programming problem and give the optimality equation. The decision variable should represent the amount of stock right after the arrival of the order and prior to the delivery.

29. Consider a deterministic dynamic minimization problem with an infinite planing horizon and discount factor $\alpha \in (0, 1)$. Let the set $\mathcal{S}$ of the feasible states and the set $\mathcal{A}$ of the feasible decision be given as $\mathcal{S} = \{0, 1, 2\}$ and $\mathcal{A} = \{0, 1, 2\}$, respectively. Moreover let $S_n = \mathcal{S}$, and $A_n(s) = \mathcal{A}$ hold for all $n \in \mathbb{N}$ and for all $s \in \mathcal{S}$. Further let the state transition function and the one state cost function be given as follows:

$$z(s, a) = (s + a) \bmod 3\,, \quad r(s, a) = \gamma a^2 - \beta(s + 1)\,, \text{ for all } s \in \mathcal{S}, \text{ and for all } a \in \mathcal{A},$$

where $\gamma > 0$, $\beta > 0$, are two given parameters. Formulate some conditions to be fulfilled by the parameters γ and β such that the policy iteration procedure terminates with an optimal solution right after the second iteration. How would you comment/interpret this result?
(Consider that α is a given constant in $(0, 1)$. The conditions on β and γ have to be specified depending on α.)