Operations Research
winter term 2019/2020

1st work sheet (integer programming)

1. As the leader of an oil-exploration drilling venture you must determine the least-cost selection of 5 out of 10 possible sites. Label the sites S_1,S_2,..., S_{10} and the corresponding exploration costs by C_i, $1 \leq i \leq 10$, respectively. Consider the following regional development restrictions.

   (i) Evaluating sites S_1 and S_7 will prevent you from exploring site S_8.

   (ii) Evaluating site S_3 or S_4 prevents you from assessing site S_5.

   (iii) Of the group S_5, S_6, S_7 and S_8 only two sites may be assessed.

   Formulate an integer programm to determine the minimum cost exploration scheme that satisfies these restrictions.

2. Three different items are to be routed through three machines. Each item must be processed first on machine 1, then on machine 2 and finally on machine 3. The sequence of items my differ for each machine. Assume that for $i, j \in \{1, 2, 3\}$ the times t_{ij} required to perform the work on item i by machine j are known and are integers. Our objective is to minimize the total time necessary to process all the items.

   (a) Formulate the problem as an integer programming problem.
   Hint: Let x_{ij} be the starting time of processing item i on machine j. Your model must prevent two items from occupying the same machine at the same time; also an item may not start processing on machine $j+1$ unless it has completed processing on machine j.

   (b) Suppose we want the items to be processed in the same sequence on each machine. Change the formulation in part (a) accordingly.

3. Consider a placement problem for firehouse locations. Assume that the population is concentrated in I districts within the city and that for each $i \in I$ district i contains p_i people. Preliminary analysis has limited the potential location of firehouses to J sites. Let d_{ij} be the distance from the center of district i to site j, for $i \in I$, $j \in J$. The goal is to determine the "best" site selection and the assignment of the districts to firehouses such that the following criteria are fulfilled.

   (i) Every district is assigned to al least one firehouse.

   (ii) It costs $f_j(s_j)$ to build a firehouse at site j to service s_j people; for each $j \in J$ the function f_j is convex and piecewise linear with two linear pieces with gradients a_j and b_j, $a_j < b_j$, respectively.

   (iii) There is a total budget B allocated for the firehouse construction.

   (iv) A central district is particularly susceptible to fire and either sites 1 and 2 or sites 3 and 4 can be used to protect this district.

   (v) The social-welfare goal is to minimize the distance travelled to the district farthest from its assigned firehouse.

   Formulate this problem as a linear integer program.

4. In the lecture we discussed the following simplified version of the warehouse location problem.

$$\text{mminimize} \qquad \sum_i \sum_j c_{ij} x_{ij} + \sum f_i y_i \,,$$

subject to

$$\sum_i x_{ij} = d_j \,, \text{ for } j = 1, 2, \ldots, n$$

$$\sum_j x_{ij} - y_i \sum_j d_j \leq 0\,, \text{ for } i = 1, 2, \ldots, m$$
$$x_{ij} \geq 0\,, \text{ for } i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n$$
$$y_i \in \{0, 1\}\,, \text{ for } i = 1, 2, \ldots, m.$$

(a) The above model assumes only one product and two distribution stages (from warehouse to customer). Indicate how the model would be modified if there were three distribution stages (from plant to warehouse to customer) and L different products.
Hint: Define a new decision variable x_{ijkl} equal to the amount to be sent from plant i, through warehouse j, to customer k, of product l.

(b) Suppose there are maximum capacities for plants and size limits, both upper and lower bounds, for warehouses. What is the relevant model now?

(c) Assume that each customer has to be served from a single warehouse, i.e., no splitting of order is allowed. How should the warehousing model be modified?

(d) Assume that each warehouse i experiences economies of scale when shipping above a certain threshold quantity to an individual customer j; i.e., the unit distribution cost is c_{ij} whenever the amount shipped is between 0 and d_{ij}, and c'_{ij} (lower than c_{ij}) whenever the amount shipped exceeds d_{ij}. Formulate the warehouse location model under these conditions.

5. Consider the following word game. You are assigned a number of tiles each containing a letter a, b, … or z from the alphabet. For any letter a from the alphabet your assignment includes N_a tiles the letter a. From the letters you are to construct any of the words w_1, …, w_n. This list might, for example, contain all words from a given dictionary. You may construct any word at most once and use any tile at most once. You receive $v_j \geq 0$ points for making a word w_j and an additional bonus $b_{ij} \geq 0$ for making both words w_i and w_j, for $i, j \in \{1, 2 \ldots, n\}$.

   (a) Formulate a linear integer program to determine your optimal choice of words.

   (b) How does the formulation change if you are allowed to select 100 tiles with no restricon on your choice of the letters?

6. Consider the problem

$$\begin{aligned} \text{maximize } z &= x_1 + 2x_2 \\ \text{subject to} & \\ & x_1 + x_2 \leq 8 \\ & -x_1 + 2x_2 \leq 2 \\ & x_1 - x_2 \leq 4 \\ & x_2 \geq 0 \text{ and } x_2 \text{ integer} \\ & x_1 \in \{0, 1, 4, 6\} \end{aligned}$$

   (a) Reformulate the problem as an integer linear program.

   (b) How would your answer to part (a) change if the objective function was changed to maximize $z = x_1^2 + 2x_2$?

7. Graph the following integer programm

$$\begin{aligned} \text{maximize } z &= x_1 + 5x_2 \\ \text{subject to} & \\ & -4x_1 + 3x_2 \leq 6 \\ & 3x_1 + 2x_2 \leq 18 \\ & x_1, x_2 \geq 0 \text{ and } x_1,\, x_2 \text{ integer} \end{aligned}$$

Apply the branch-and-bound procedure, graphically solving each linear programming encountered. Interpret the branch-and-bound procedure graphically as well.