Combinatorial optimization 1
Winter term 2016/2017

6th working sheet

37. Consider a bipartite graph $G = (U\dot{\cup}V, E)$ with $U = \{u_1, u_2, u_3, u_4, u_5\}$, $V = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set specified by the matrix A below:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

The row indices of A corresponds to the indices of vertices in U, the column indices of A correspond to the vertex indices in V, and an entry a_{ij} equals 1 if and only if there is an edge (u_i, v_j) in $E(G)$. Determine algorithmically a matching with maximum cardinality in G. Use that matching to determine a minimum vertex cover in G.

38. Let $r \in \mathbb{N}$ and let $G = (A\dot{\cup}B, E)$ be an r-regular bipartite graph, i.e. a bipartite graph for which $\deg(v) = r$, $\forall v \in V(G)$. Show that the edge set $E(G)$ can be partitioned into r disjoint perfect matchings.

39. Prove the Tutte-Berge formula for the maximum cardinality of a matching in a graph G

$$\nu(G) = \frac{1}{2} \min \{|V(G)| - q(G \setminus A) + |A| : A \subseteq V(G)\}.$$

$\nu(G)$ is the matching number of G and $q(G \setminus A)$ is the number of odd connected components in the graph $G \setminus A$ obtained from G by removing all vertices in A and all edges adjacent to any of the removed vertices.

Hint: You could use the blossom algorithm of Edmonds to obtain an algorithmic proof.

40. Use Edmonds blossom algorithm to determine a matching with maximum cardinality in the graph given in Figure 1. Initialise the algorithm with the empty matching. Specify a subset A of the vertex set $V(G)$ for which the minimum in the Tutte-Berge formula is reached and justify your choice.

41. Show that every (simple) graph G with n vertices and minimum degree $\delta(G) = k$, i.e. $k := \min\{deg(v) : v \in V(G)\}$, has a matching with cardinality $\min\{\lfloor \frac{n}{2} \rfloor, k\}$. (A simple graph is a graph without multiple edges and without loops.)

Hint: Use the Tutte-Berge formula.

42. Show that a 3-regular graph with at most two bridges has a perfect matching. A bridge in a graph G is an edge $e \in E(G)$ with the property that the graph $G - e$ obtained from G by removing edge e has more connected components than G. Is there any 3-regular graph which does not have a perfect matching?

Hint: Use the Tutte-Berge formula.

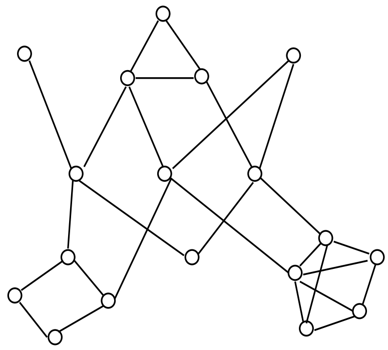

Figure 1: Graph for exercise no. 40.