Combinatorial optimization 1
Winter term 2016/2017

3d working sheet[1]

21. Let $G = (V, E)$ be a digraph with conservative weights $c : E \to \mathbb{R}$. Give an efficient algorithm to compute a (directed) cycle with smallest weight in G.

    Hint: Modify the all-pairs-shortest-path algorithm of Floyd und Warshall.

22. Consider the digraph in Figure 1 and diminish the weight of every edge by 2.

    (a) Determine whether this graph contains a cycle with negative weight (by applying an efficient algorithm, not by complete enumeration).

    (b) Determine algorithmically a cycle with minimum mean weight.

23. (Metric closure)
    Let a graph G (directed or undirected) with conservative weights $c\colon E(G) \to \mathbb{R}$ be given. The *metric closure* of (G, c) is the pair $(\bar{G}, \bar{c})$ specified as follows. $\bar{G}$ is a (directed resp. undirected) graph such that $V(\bar{G}) = V(G)$, and for any $x, y \in V(G)$, $x \neq y$, $(x, y) \in E(\bar{G})$ holds iff there exists a (directed resp. undirected) x-y-path in G. The weights $\bar{c}$ are the lengths of the corresponding shortest paths in G, i.e. $\bar{c}((x, y)) = \mathrm{dist}_G(x, y)$, for any $(x, y) \in E(\bar{G})$.

    Let G be a complete (undirected) graph and $c\colon E(G) \to \mathbb{R}_+$. Show that (G, c) is its own metric closure iff the triangle inequality holds, i.e. $c((x, y)) + c((y, z)) \geq c((x, z))$, for any three pairwise distinct vertices x, y, z in $V(G)$.

24.* The timing constraints of a logic chip can be modelled by a digraph G with edge weights $c\colon E(G) \to \mathbb{R}_+$. The vertices represent the storage elements, the edges represent paths through combinational logic, and the weights are worst-case estimations of the propagation time of a signal. An important task in the design of the very large scale integrated (VLSI) circuits is to find an optimum clock schedule, i.e. a mapping $a\colon V(G) \to \mathbb{R}$ such that $a(v) + c((v, w)) \leq a(w) + T$ for all $(v, w) \in E(G)$ and a number T which is as small as possible. (T is the cycle time of the chip; $a(v)$, and $a(v) + T$ are the "departure time" and the latest feasible "arrival time" of a signal at v, respectively).

    (a) Reduce the problem of finding the optimum T to the (directed) minimum mean cycle problem.

    (b) Show how the numbers $a(v)$ of an optimum solution can be determined efficiently.

    (c) Typically some of the numbers $a(v)$ are fixed in advance. Show how to solve the problem in this case.

25. Use the maximum-flow algorithm of Ford and Fulkerson to determine the maximum flow and its value in the network in Figure 2, where the numbers on the edges are the capacities. Start with the initial flow given below:

Kante	s-1	s-2	s-3	1-2	2-3	1-t	2-t	3-t
Fluss	8	8	0	2	2	6	8	2

    Extend the algorithm so that it also computes a minimum s-t-cut.

26. Let (G, u, s, t) be a network, and let $\delta^+(X)$ and $\delta^+(Y)$ be minimum s-t-cuts in (G, u). Show that $\delta^+(X \cap Y)$ and $\delta^+(X \cup Y)$ are also minimum s-t-cuts in (G, u).

[1] Exercises marked with an $*$ are more demanding than others.

27. Prove Hoffman's circulation theorem: Given a digraph G and lower and upper capacities $l, u \colon E(G) \to \mathbb{R}_+$ with $l(e) \leq u(e)$ for all $e \in E(G)$, respectively, then there is a circulation f with $l(e) \leq f(e) \leq u(e)$ for all $e \in E(G)$, iff

$$\sum_{e \in \delta^-(X)} l(e) \leq \sum_{e \in \delta^+(X)} u(e) \text{ for all } X \subseteq V(G)\,.$$

(Recall the definition of a circulation as a flow with value 0.)
Notice that Hoffman's circulation theorem implies easily the max-flow-min-cut theorem.

28.* Convergence of the Ford-and-Fulkerson-algorithm for the mamximum flow problem.

(a) Show that the algorithm of Ford and Fulkerson does not necessarily terminate in the case of irrational capacities. To this end you can consider the network in Figure 3. Each line segments represents two edges (arcs) in both directions. Every edge has capazity $\frac{1}{1-\sigma}$ except for the four edges listed below together with their capacities:

$$u((x_1, y_1)) = 1\,,\ u((x_2, y_2)) = \sigma\,,\ u((x_3, y_3)) = u((x_4, y_4)) = \sigma^2\,,$$

where $\sigma := \frac{\sqrt{5}-1}{2}$. Observe that $\sigma^n = \sigma^{n+1} + \sigma^{n+2}$ holds. Investigate the convergence of the sequence $(v(f_i))_{i \in \mathbb{N}}$ where $v(f_i)$ is the value of the flow computed in i-th iteration of the Ford-and-Fulkerson-algorithm

(b) Solve the above problem with the algorithm of Edmonds und Karp.

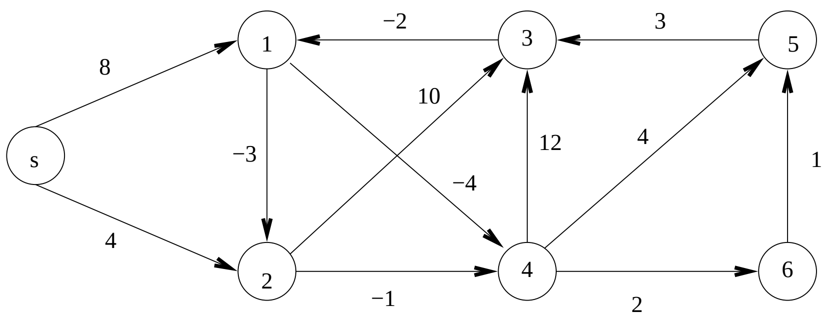

Figure 1: Digraph for Exercise 22

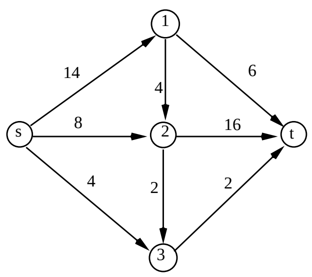

Figure 2: Network for Exercise 25

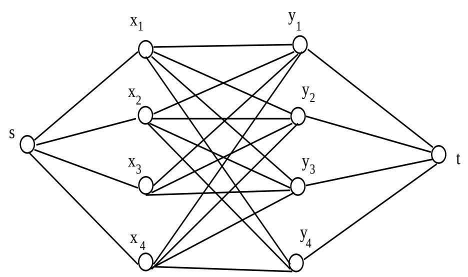

Figure 3: Network for Exercise 28