Game Theory, Summer Term 2018

Exercise Sheet 5

30. (Strategic equivalence and monotonicity)

    (a) In some sources of literature the coalitional function v (which is also called the characteristic function) of a coalitional game (N, v) is just required to fulfill $v(\emptyset) = 0$ but not necessarily $v(S) \leq v(T)$ for any $S, T \subseteq N$ with $S \subseteq T$, as required in the definition given in the lecture. Coalitional functions for which the above monotonicity property is fulfilled are then called *monotonic*. We will adopt these definitions in this exercise and in the next one.

    Consider a three players game with $N = \{1, 2, 3\}$ and $v(1) = 3$, $v(2) = 13$, $v(3) = 4$, $v(1, 2) = 12$, $v(1, 3) = 15$, $v(2, 3) = 1$, $v(1, 2, 3) = 10$. Is this game monotonic[1]?

    (b) A coalitional game (N, w) is *strategically equivalent* to the game (N, v), iff there exists a positive number a and a vector $b \in \mathbb{R}^{|N|}$, such that for every coalition $S \subseteq N$, $w(S) = av(S) + \sum_{i \in S} b_i = av(S) + b(S)$, for $S \subseteq N$, where $b(S)$ is defined by $b(S) := \sum_{i \in S} b_i$.

    Find a monotonic game which is strategically equivalent to the game (N, v) given in (a).

    (c) Prove that every coalitional game is strategically equivalent to a monotonic game.

31. (Strategic equivalence and the core)

    Show that the core of a coalitional game is *covariant* under strategic equivalence (see Exercise no. 30), i.e.

    $$\mathcal{C}(N, av + b) = a\mathcal{C}(N, v) + b\,,$$

    holds for any $a \in \mathbb{R}$, $a > 0$, and any $b \in \mathbb{R}^n$. $\mathcal{C}(N, v)$ denotes here the core of the game (N, v) (cf. the definition given in the lecture) and $aX + b$ is defined as $aX + b := \{ax + b \colon x \in X\}$, for any $X \subseteq \mathbb{R}^{|N|}$, any $a > 0$, and any $b \in \mathbb{R}^{|N|}$.

32. (Weighted majority games) A coalitonal game (N, v) is called a *simple game* if $v(S) \in \{0, 1\}$, for any $S \subseteq N$. A coalition S with $v(S) = 1$ is called a *winning coalition*, a coalition S with $v(S) = 0$ is called a *losing coalition*. A *weighted majority game* (N, v) is a special case of a simple game where we are given a quota q and nonnegative weights w_i for each player $i \in N$, such that the value of any coalition $S \subseteq N$ is given as

    $$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q, \\ 0 & \text{otherwise.} \end{cases}$$

    Such a weighted majority game is denoted by $[q, w_1, w_2, \ldots, w_N]$.

    (a) Check that the following weighted majority games share the same coalitional function and therefore are different representations of the same game:

    $$[2; 1, 1, 1]\,,\ [9; 8, 2, 7]\,,\ [9; 8, 1, 8].$$

    (b) The representation $[2; 1, 1, 1]$ from (a) has the property that the sum of the weights equals the quota 2 in every minimal winning coalition. (A minimal winning coalition is a winning coalition such that every one of its proper subsets is not winning.) Such weights are called *homogeneous weights*, and the corresponding representation of the

[1] Notice that for the ease of notation we have omitted here the curly parentheses which should sorround any subset of N given as a list of its elements in the specifications of v.

game is called a *homogeneous representation*. In general the weights $w_1, w_2, \ldots, w_n$ are called *homogeneous weights*, iff there exists a real number q such that in the weighted majority game $[q; w_1, \ldots, w_n]$ the equality $q = \sum_{i \in S} w_i$ holds for every minimal winning coalition. $[q; w_1, \ldots, w_n]$ is then called a *homogeneous representation*.

For each of the two following weighted majority games determine whether it has a homogeneous representation. If yes, write it down. If no, explain why:

$$[10; 9, 1, 2, 3, 4], \quad [8; 5, 4, 2].$$

33. A player i in a simple game is a *veto player*, iff $v(S) = 0$ for every coalition S that does not contain i.

    (a) Show that the core of a simple game (N, v) satisfying $v(N) = 1$ contains every allocation vector $\psi \in \mathbb{R}^{|N|}$ satisfying $\psi_i = 0$ for every player i which is not a veto player, and does not contain any other allocation vectors. In other words, the only allocation vectors in the core are those in which the set of veto players divide the worth $v(N)$ of the *grand coalition* N between them.

    (b) Consider a simple majority game (N, v) with $n := |N|$ in which a coalition wins if and only if it has at least $\frac{n+1}{2}$ votes; that is, for every coalition $S \subseteq N$, the following equality holds:

    $$v(S) = \begin{cases} 1 & \text{if } |S| \geq \frac{n+1}{2}, \\ 0 & \text{if } |S| < \frac{n+1}{2}. \end{cases}$$

    Determine the core of this game.

    (c) Determine the core of a simple coalitional game without veto players.

34. (The glove market revisited)

    A proper pair of gloves consists of a left glove and a right glove. There are n players. Player 1 hast two left gloves, while each of the other $n-1$ players has one right glove. The payoff $v(S)$ for a coalition S is the number of proper pairs that can be formed from the gloves owned by the members of S.

    (a) For $n = 3$ determine $v(S)$ for each of the seven nonempty sets $S \subseteq \{1, 2, 3\}$. Then find the Shapley value $\psi_i(v)$ for each of the players.

    (b) For a general n find the Shapley value $\psi_i(v)$ for each of the players $i = 1, 2, \ldots, n$.

35. Let ϕ_1 and ϕ_2 be two solution concepts for the family $\mathcal{F}$ of bargaining games (cf. the definition given in the lecture). Define another solution concept ϕ for the family $\mathcal{F}$ of bargaining games as follows:

    $$\phi(S, d) = \frac{1}{2}\phi_1(S, d) + \frac{1}{2}\phi_2(S, d), \text{ for any } (S, d) \in \mathcal{F}.$$

    For a property P prove or disprove the claim: if both ϕ_1 and ϕ_2 satisfy the property P, then ϕ also satisfies the same property P, where

    $$P \in \{\text{symmetry, Pareto optimality, affine covariance, independence of irrelevant alternatives}\}.$$

36. Two players are to divide 2000€ between them. The utility functions of the first and the second player are $u_1(x) = x$ and $u_2(s) = \sqrt{x}$, respectively. For each of the following two situations, describe the corresponding bargaining game in utility units, and find its Nash solution.

    (a) If the two players cannot come to an agreement, neither of them recieves any payoff.

    (b) If the two players cannot come to an agreement, the first one recieives 16€ and the second one receives 49€ (not that in this case the disagreement point in the utility space is $(16, 7)$).