Game Theory, Summer Term 2018

Exercise Sheet 4

23. Let $c > 0$ and let f_c be a function defined as follows

$$f_c(x) = \begin{cases} \frac{1}{c-x} & \text{if } 0 \leq x < c \\ \infty & \text{otherwise} \end{cases} .$$

Consider a selfisch routing network where all latency functions are in $\mathcal{L}$ with $\mathcal{L} := \{f_c(\cdot)\colon c > 0\}$. For every edge e in the network let its latency function be $f_{c_e}(x)$ for some $c_e > 0$. Supose moreover that in equilibrium, the edge flow F_e satisfies $F_e \leq (1-\beta)c_e$ for every edge e, where $\beta \in (0,1)$ is a given constant. Derive a finite upper bound for the price of anarchy in this case.

24. Let $G = (V, E)$ be a selfish routing network where r units of traffic are routed from origin s to destination t, with $s, t \in V$. Suppose that $l_e(\cdot)$ is the latency function associated with edge $e \in E$. Consider now a selfisch routing instance with the same network G, the same origin s and the same destination t, as well as the same amount r of flow to be routed, and the latency function is $l'_e(\cdot)$, is given as

$$l'_e(x) := \frac{l_e(x/2)}{2}\,, \ \forall e \in E.$$

Suppose that f^* is an optimal flow in the original instance and f' is an equilibrium flow in the instance with modified latency functions. Prove that $L(f') \leq L(f^*)$, where $L(f')$ is the total latency of f' in the modified instance and $L(f^*)$ is the total latency of f^* in the original instance.

25. Show that the price of anarchy bound for the market sharing game considered in the lecture and also in Exercise No. 17 can be improved to $2 - 1/k$, when there are k teams. Show that this bound is tight.

26. Consider atomic selfish routing in a Pigou network (cf. the lecture) where 2 drivers want to travel from the origin to the destination. Let the latency function on the top edge be given by x, where x is the number of drivers using that edge, and let the latency function be a constant equal to 2 on the bottom edge. What is the total latency for the optimal routing? Find all Nash equilibria for this instance of the atomic selfish routing game and determine the price of anarchy.

27. Consider the following network formation game. There are n vertices each representing a player. The pure strategies of a player consist of choosing which other vertices to create an edge to. A strategy profile induces then a graph where each edge is associated with the vertex that "created" it. Given a strategy profile s the cost incurred by player i is

$$cost_i(s) := \alpha \cdot n_i(s_i) + \sum_{j \neq i} d_s(i,j)\,,$$

where $n_i(s_i)$ is the number of links created by vertex i (each link costs α to create), $d_s(i,j)$ is the *distance* from i to j in the graph resulting from strategy profile s (i.e. the number of edges of an i-j-path with the smallest possible number of edges), and $\alpha > 0$ is a constant representing the cost of setting up an edge.

(a) Show that if $\alpha \geq 2$, then the graph which minimizes $\sum_i cost_i$ is a star, whereas when $\alpha < 2$ then it is a complete graph[1]. The graph which minimizes the total cost as above is called the *optimum graph*.

(b) Show that if $\alpha \leq 1$ or $\alpha \geq 2$, there is a Nash equilibrium with total cost equal to that of the optimum graph.

(c) Show that for $1 < \alpha < 2$ there is a Nash equilibrium with total cost at most 4/3 that of the optimum graph.

28. (Evolutionary stable strategies)

Consider a two-player symmetric game. A mixed strategy $x^* \in \mathbb{R}^2$ in a two player symmteric game is called an *evolutionary stable strategy* (ESS) if for every mixed strategy $x \in \mathbb{R}^2$ that differs from x^*, there exists an $\epsilon_0 := \epsilon_0(x) > 0$ such that the following holds:

$$\forall \epsilon \in (0, \epsilon_0),\ (1-\epsilon)u_1(x, x^*) + \epsilon u_1(x, x) < (1-\epsilon)u_1(x^*, x^*) + \epsilon u_1(x^*, x,)\,.$$

This definition is motivated by biological applications and its interpretation is the following. Consider a population mostly composed of "normal" individuals with a minority of mutations. Interprete x^* as the distribution of behavior types among the normal individuals and x as the distribution of behavior types in a mutation, where the proportion of this mutation in the population is ϵ. Every mutated individual will encounter a normal individual with probability $(1-\epsilon)$ and a mutated individual with probability ϵ, obtaining thereout a payoff of $u_1(x, x^*)$ and $u_1(x, x)$, respectively. So the above inequality says that the expected payoff of a mutated individual is smaller than the expected payoff of a normal individual, and hence the proportion of mutations will decrease and eventually disappear over time, with the type behavior of the population remaining mostly x^*.

Observe that if x^* is an ESS in a two-player symmetric game, then (x^*, x^*) is a symmetric Nash equilibrium in this game. Show that a strategy x^* is an ESS if and only if for each $x \neq x^*$ only one of the following two conditions hold:

$$u_1(x, x^*) < u_1(x^*, x^*) \text{ or}$$

$$u_1(x, x^*) = u_1(x^*, x^*) \text{ and } u_1(x, x) \leq u_1(x^*, x)\,.$$

Conclude from the above conditions that if (x^*, x^*) is a *strict symmetric Nash equilibrium* in a symmetric game, than x^* is an ESS. Recall that a strict symmetric Nash equilibrium is a Nash equilibrium for which any unilateral deviation leads to a deterioration of the utility of the deviating player.

29. Suppose that a particular animal species can exhibit one of the two possible behaviours: agressive behaviour or peaceful behaviour. We call this behaviours Hawk (behaviour) and Dove (behaviour), respectively. The two types of behaviour are expressed when an animal invades the territory of an other animal of the same species. Consider two different species and let the payoff matrices in the cases (a) and (b) below describe the expected number of offsprings of each type of animal for each species, respectively. Find all Nash equilibria of the game and also an ESS in both cases.

(a)

		Defender: Dove	Defender: Hawk
	Dove	(4,4)	(2,8)
Invader	Hawk	(8,2)	(1,1)

(b)

		Defender: Dove	Defender: Hawk
	Dove	(4,4)	(1,3)
Invader	Hawk	(3,1)	(2,2)

[1] A *star* is a graph $G = (V, E)$, where there is a particular vertex $v_0 \in V$, called the *central vertex*, such that $E = \{(v_0, v) : v \in V \setminus \{v_0\}\}$. A *complete graph* is a graph in which every pair of vertices is connected by an edge.