Game Theory, Summer Term 2018

Exercise Sheet 3

16. A simultaneous congestion game.

    There are two drivers, one who will travel from A to C, the other from B to D, see Figure 1. Each road is labelled (x, y) where x is the cost to any driver who travels the road alone and y is the cost to each driver if both drivers use this road. Write the game in matrix form and find all pure Nash equiibria.

17. A market sharing game.

    There are k NBA teams, and each of them must decide in which city to locate. Let $n \in \mathbb{N}$ and $C = \{1, 2, \ldots, n\}$ be the set of possible locations (cities). Let v_j be the profit potential, e.g. the number of basketball fans, of city j. If l teams select city j than each obtains a utility of v_j/l. Let $c = (c_1, \ldots, c_k)$ denote a strategy profile, where c_i is the city selected by team i, and let $n_{c_i}(c)$ be the number of teams that select city c_i in this profile, for $1 \le i \le k$. Show that the market sharing game is a potential game with potential function

    $$\Phi(c) = \sum_{j \in C} \sum_{l=1}^{n_{c_i}(c)} \frac{v_j}{l}$$

    and hence has a pure Nash equilibrium.

18. Consider the following variant of the Consensus game (cf. the lecture). Let $G = (V, E)$ be an arbitrary undirected graph where each vertex $i \in V := \{1, 2, \ldots, n\}$ is a player and her action consists of choosing a bit in $\{0, 1\}$. Let vertex i's choice be represented by $b_i \in \{0, 1\}$, for $i \in V$, and write $b = (b_1, \ldots, b_n)$ for the corresponding strategy profile. Let $N(i)$ be the set of neighbors of i in G, for all $i \in V$. Consider a weight w_{ij} on each edge $\{i, j\}$ which measures how much the two players i and j care about agreeing with each other, for all $\{i, j\} \in E$ (since G is an undirected graph graphs we assume that $w_{ij} = w_{ji}$, for all $\{i, j\} \in E$). The loss $D_i(b)$ for player i under strategy profile b is the total weight of neighbors that she disagrees with, i.e.

    $$D_i(b) = \sum_{j \in N(i)} |b_i - b_j| w_{ij} \, .$$

    Show that this variant of the Congestion game is a potential game.

    Consider now a slightly different version of the above game played on a directed graph with weight w_{ij} which are not necessarily symmetric, i.e. in general $w_{ij} \neq w_{ji}$ can hold for $\{i, j\} \in E$. Show that in general this variant of the game is not a potential game.

19. Construct an example showing that the Graph Coloring game (c.f. the lecture) has a Nash equilibrium which uses more than $\chi(G)$ colors.

20. The definition of a potential game extends naturally to k player games with infinite strategy spaces S_i as follows. Call $\psi : \Pi_{i=1}^{k} S_i \to \mathbb{R}$ a potential function if for all players i the function $s_i \mapsto \psi(s_i, s_{-i}) - u_i(s_i, s_{-i})$ is constant on S_i. Show that the game where the k players send data along a shared channel of capacity 1, as discussed in the lecture, is a potential game.

    Hint: Consider the case of 2 players with strategies $x, y \in [0, 1]$. Then there must exist a c_x depending just on x and a c_y depending just on y such that $\psi(x, y) = c_y + x(1 - x - y) = c_x + y(1 - x - y)$, i.e. $c_y + x(1 - x) = c_x + y(1 - y)$, for $x, y \in [0, 1]$.

21. Infinite strategy spaces: Club Pricing.

    Three neighboring colleges have n students each that hit two clubs $C1$ or $C2$ on weekends. Each of the two clubs, which are the players, chooses an entry price in $[0, 1]$. College A students go to $C1$, College C students go to C_2 and College 3 students choose to go to the club with the lowest price that weekend, breaking ties in favor of $C1$. Let the pure strategies of $C1$ and $C2$ be described by $p_1 \in [0, 1]$ and $p_2 \in [0, 1]$, respectively. Write the utility functions of the two players (by distinguishing the cases $p_1 \leq p_2$ and $p_1 > p_2$). Show that there are no pure Nash equilibria in this game. Show however that there is a symmetric mixed Nash equilibrium (F, F), where F is a continuous distribution function on $[0, 1]$, i.e. F is a best response of $C1$ ($C2$) with respect to its expected payoff provided that the other player $C2$ ($C1$) chooses its price according to distribution F.

    Hint: Show by domination that w.l.o.g. the support of a mixed Nash equilibrium strategy (which is a probability distribution on $[0, 1]$) can be assumed to be equal to $[1/2, 1]$.

22. Price of anarchy.

    Let $G = (V, E)$ be a (directed) network where one unit of traffic is routed from a source s to a destination t. Suppose that the latency function on each edge is linear, i.e. $l_e(x) = a_e x$, for constants $a_e \geq 0$, and for all $e \in E$. Show that the price of anarchy of such a network equals 1.

    Hint: Appropriately modify the proof of the analogous result for affine functions (cf. the lecture) and use the inequality $xy \leq (x^2 + y^2)/2$ therein.

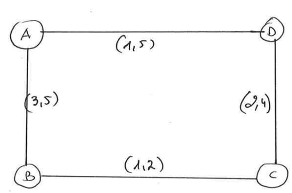

Figure 1: Road network for Exercise No. 16