Advanced and algorithmic graph theory
Summer term 2020

Sixth worksheet

51. Determine the Ramsey number $R(3)$.

52. Show that for every tree T on $t \geq 2$ vertices and for any $s \geq 2$, $s, t \in \mathbb{N}$, the equality $R(T, K_s) = (s-1)(t-1)+1$ holds. This results is due to Chvátal (1977) [2] and a proof can also be found in [1]. Recall that $R(T, K_s)$ is the smallest natural number $n \in \mathbb{N}$ such that for every graph G of order n either T is a subgraph of G or the complete graph K_s on s vertices is a subgraph of the complement $\bar{G}$.

53. Can you improve on the exponential upper bound on the Ramsey number $R(n)$ obtained in Theorem 7.3 (cf. the lecture) in the case of perfect graphs?

54. For the graphs F and H shown in Figure 1 determine $R(F, H)$, i.e. the smallest natural number $n \in \mathbb{N}$ such that for every graph G of order n either F is a subgraph of G or H is a subgraph of the complement $\bar{G}$.

55. (a) What is the probability that a random graph in $\mathcal{G}(n, p)$ has exactly m edges, for fixed $m \in \mathbb{N}$, $0 \leq m \leq \binom{n}{2}$ fixed?
    (b) What is the expected number of edges in $G \in \mathcal{G}(n, p)$?
    (c) What is the expected number of subgraphs of $G \in \mathcal{G}(n, p)$ which are isomorphic to K_r, for $1 \leq r \leq n$?

56. Prove or disprove:
    (a) For each constant $p \in (0, 1)$ almost no graph in $\mathcal{G}(n, p)$ is planar (as n tends to infinity).
    (b) For each natural number $k \in \mathbb{N}$ and each constant $p \in (0, 1)$ almost no graph in $\mathcal{G}(n, p)$ is k-colorable (as n tends to infinity).

57. Show that for each constant $p \in (0, 1)$ almost every graph in $\mathcal{G}(n, p)$ has diameter 2 (as n tends to infinity).

58. For every $k \geq 1$, $k \in \mathbb{N}$, find a threshold function for $\{G \colon \Delta(G) \geq k\}$.

59. For every $d \in \mathbb{N}$ determine the threshold function for the property of containing the d-dimensional cube (as defined in Exercise 2) and for the property of containing the complete graph K_d.

60. Does the property of containing any tree of order k, for some fixed $k \in \mathbb{N}$, have a threshold function? If yes, which? If not, why not?

61. Show that $t(n) = \frac{1}{n}$ is a threshold function for the property of containing any cycle.

    Hint: For $p/t \to 0$ apply Markov's inequality and Lemma 7.7 about the expected number of cycles on k vertices; for $p/t \to \infty$ apply Corollary 7.18 (cf. the lecture).

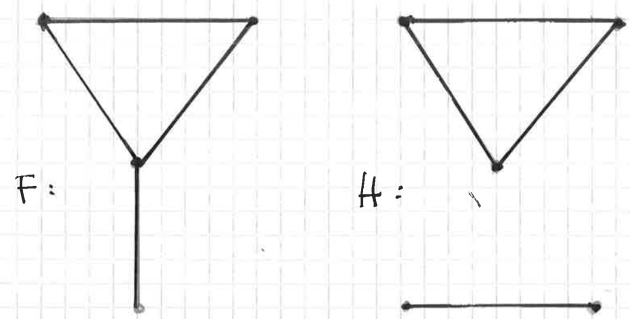

Figure 1: Graph for Exercise 54

References

[1] G. Chartrand, L. Lesniak and P. Zhang, *Graphs and Digraphs*, CRC Press, Taylor and Francis Group, 2016.

[2] V. Chvátal, Tree-complete Ramsey numbers, *Journal of Grpah Theory* **1**, 1977, 93.