Advanced and algorithmic graph theory
Summer term 2020

Fourth worksheet

30. This example shows that the clique number $\omega(G)$ can be an arbitrarily bad lower bound on the chromatic number $\chi(G)$ of a graph G.

    Consider the sequence of graphs M_k, $k \in \mathbb{N}$, $k \geq 3$, constructed recursively as follows (cf. Mycielski 1955 [3]). Start with $M_3 := C_5$, the cycle on 5 vertices. The graph M_{k+1} is obtained from M_k by adding to M_k $(n+1)$ new vertices $u_1, u_2, \ldots, u_n, w$, where $n := |V(M_k)|$, such that w is connected to each u_i, $1 \leq i \leq n$, and u_i is connected to all vertices in $\Gamma(v_i)$, i.e. to all neighbours of v_i, $1 \leq i \leq n$. Show the following properties of M_k, $k \geq 3$:

    (i) M_k is triangle-free, i.e. it contains no cycle of length 3, $\forall k \geq 3$,

    (ii) $\chi(M_k) = k$,

    (iii) $|V(M_k)| = 3 \cdot 2^{k-2} - 1$.

    The graph M_4 is also called the Grötzsch-Graph [1].

31. This example shows that the quotient $\frac{|V(G)|}{\alpha(G)}$ can be an arbitrarily bad lower bound on the chromatic number $\chi(G)$ of a graph G with stability number $\alpha(G)$.

    Let G_k be a graph with $2k+1$ vertices such that $V(G_k) = V(K_k) \dot{\bigcup} V(S_k) \dot{\bigcup} \{w\}$, where K_k is an induced subgraph of G_k which is complete and has k vertices, S_k is an induced subgraph of G_k which has no edges and k vertices, and w is a vertex in G_k connected to all vertices of S_k and to no vertex of K_k. Moreover each vertex of S_k is connected to all vertices of K_k. Show that $\chi(G_k) = k+1$ and $\alpha(G_k) = k$ and deduce thereout the arbitrarily bad quality of the bound $\frac{|V(G)|}{\alpha(G)}$ for $\chi(G)$.

32. Calculate the chromatic number of a graph in terms of the chromatic numbers of its blocks.

33. (a) Show that every graph G has a vertex ordering for which the greedy algorithm only uses $\chi(G)$ colours.

    (b) For every $n \in \mathbb{N}$, $n \geq 2$, find a bipartite graph on $2n$ vertices ordered in such a way that the greedy algorithm uses n rather than 2 colours.

34. Find a graph G for which Brooks theorem yields a significantly weaker bound on $\chi(G)$ than the colouring number $col(G) := \max\{\delta(H) \colon H \subseteq G\} + 1$ (cf. lecture).

35. Prove or disprove: Every Hamiltonian connected graph of order at least 3 has chromatic number at least 3.

36. A *balanced coloring of a graph* G is an assignment of colors $k \in \mathbb{N}$ to the vertices of G such that (i) every two adjacent vertices are assigned different colors and (ii) the cardinalities of any two different color classes differ by at most 1. Recall that for each color k the corresponding color class C_k is defined as the set of all vertices of G colored by color k. The smallest number of colors used in a balanced coloring of a graph G is called *the balanced chromatic number of* G and is denoted by $\chi_b(G)$.

[1] M_4 is the unique smallest triangle-free 4-chromatic graph, where “smallest” refers to the number of vertices [2].

(a) Prove that the balanced chromatic number is well defined for every graph G.

(b) Determine $\chi_b(G)$ for a tree G with vertex set $V(G) = \{s, t, u, v, w, x, y, z\}$ and edge set $E(G) = \{\{v, s\}, \{v, t\}, \{v, u\}, \{v, w\}, \{w, x\}, \{w, y\}, \{y, z\}\}$.

37. A graph G is called *color-critical* if $\chi(H) < \chi(G)$ for every proper induced subgraph H of G. If G is a color-critical k-chromatic graph, then G is called critically k-chromatic or simply k-critical. Observe that K_2 is the only 2-critical graph and K_n is n-critical for every $n \in \mathbb{N}$.

(a) Show that the odd cycles are the only 3-critical graphs.

(b) Determine all k-critical graphs with $k \geq 3$ and having the property that $G - v$ is $(k-1)$-critical for every $v \in V(G)$.

(c) It is known that a k-critical graph, $k \in \mathbb{N}$, $k \geq 2$, is $(k-1)$-edge-connected. (A proof of this statement can be found e.g. in [1], Theorem 14.13 in Chapter 14.) Use this result to show that (i) $\chi(G) \leq 1 + \lambda(G)$ for every k-critical graph ($\lambda(G)$ being the edge-connectivity of G) and (ii) $\chi(G) \leq col'(G) := 1 + \max\{\lambda(H)\colon H \text{ is a subgraph of } G\}$ for every graph G. Observe that (i) implies $\chi(G) \leq 1 + \delta(G)$ for color-critical graphs G.

(d) Specify $col'(G)$ if G is (i) a tree or (ii) an outerplanar graph[2].

38. Recall from the lecture the following concepts. Given a graph G and lists of colours $L(v)$ for all $v \in V(G)$, a *vertex list colouring* is a mapping

$$c\colon V(G) \to \bigcup_{v \in V(G)} L(v)\,, v \mapsto c(v)\,,$$

such that $c(v) \in L(v)$, $\forall v \in V(G)$, and $c(u) \neq c(v)$ whenever $u, v \in V(G)$ and $\{u, v\} \in E(G)$ holds. G is called *vertex k-choosable* iff for any collection of lists $L(v)$, $v \in V(G)$, with $|L(v)| \geq k$, $\forall v \in V(G)$, there exists a vertex list colouring. The smallest natural number k for which a graph G is vertex k-choosable is called the *list chromatic number of G* (or *the choice number of G*) and is denoted by $\chi_l(G)$. Show that every plane graph is vertex 6-choosable[3].

39. For every natural number $k \in \mathbb{N}$ find a graph G with $\chi(G) = 2$ and $\chi_l(G) \geq k$.

40. Applications of coloring problems

(a) Figure 1 shows traffic lanes $L_1, \ldots, L_7$ at the intersection of two streets. A traffic light is located at the intersection. During a certain phase of the traffic light those cars in lanes for which the light is green may proceed safely through the intersection in permissible directions. What is the minimum number of phases needed for the traffic light so that eventually all cars may proceed through the intersection? A phase of a traffic light may be seen as a time interval where the colors of all lights in the traffic light do not change. Hint: You might want to consider conflicting lanes and build a graph model for them.

(b) Model the classical sudoku puzzle as a graph coloring problem. The puzzle consists of a 9×9 grid in which some of the cells hold a natural number between 1 and 9. The cells are grouped in the so called *blocks*, where a block is a 3×3 subgrid such that the set of the subgrid rows and the set of subgrid columns coincides with one of the three subsets of rows or columns $\{1, 2, 3\}$ or $\{4.5.6\}$ or $\{7, 8, 9\}$.

The puzzle consists in filling the empty cells with natural numbers between 1 and 9 such that every number between 1 and 9 appears exactly once in each row, in each column and in each block. The order in which the cells are filled is irrelevant.

[2] A graph is called outerplanar if it is planar and it can be embedded in the plane such that all of its vertices lie on the border of the outer face.

[3] There is a stronger result of Thomassen [4] stating that every planar graph is vertex 5-choosable.

References

[1] G. Chartrand, L. Lesniak and P. Zhang, *Graphs and Digraphs*, CRC Press, Taylor and Francis Group, 2016.

[2] V. Chvátal, The minimality of the Mycielski graph, in *Graphs and Combinatorics*, *Lecture Notes in Mathematics* **406**, 243–246, Springer, Berlin, 1973.

[3] J. Mycielski, Sur le coloriage des graphes, *Colloq. Math.* **3**, 1955, 161–162.

[4] C. Thomassen, Every planar graph is 5-choosable, *Journal of Combinatorial Theory (B)* **62(1)**, 1994, 180–181.

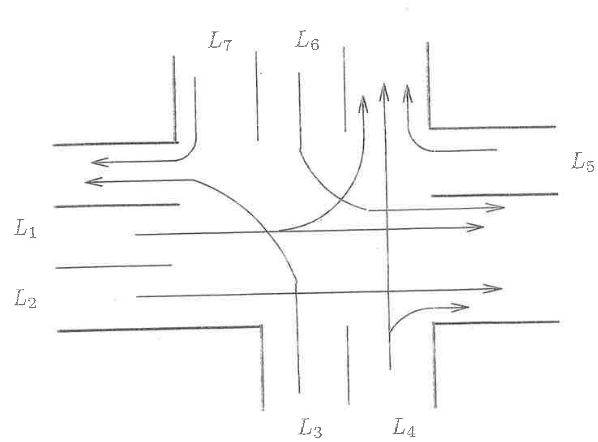

Figure 1: Graph for Exercise 40(a)