

Advanced and algorithmic graph theory
Summer term 2020

Third worksheet

22. For a nonnegative real number t , a noncomplete graph G is called t -tough if $\frac{|S|}{c(G-S)} \geq t$ holds for any nonempty separator S , i.e. for every nonempty proper subset of the vertex set $V(G)$ for which $G - S$ is disconnected. Here $c(G - S)$ denotes the number of connected components of $G - S$. The maximum rational number t for which a noncomplete graph G is t -tough is called *the toughness* of the graph G and is denoted by $t(G)$ ¹.

(a) Observe that 1-toughness is a necessary condition for the Hamiltonicity of a graph and show that this condition is not sufficient. (You might want to consider the Petersen graph.)

(b) Observe that $t(G) = 0$ if and only if G is disconnected and show that the following inequality holds

$$\frac{\kappa(G)}{\alpha(G)} \leq t(G) \leq \frac{\kappa(G)}{2}.$$

(c) Determine the toughness of the regular complete 3-partite graph $K_{r,r,r}$ for $r \in \mathbb{N}$, $r \geq 2$. $K_{r,r,r}$ is defined as the graph with vertex set $V(K_{r,r,r}) = U \dot{\cup} V \dot{\cup} W$ where $|V| = |U| = |W| = r$ and edge set $E(K_{r,r,r}) = (V \times U) \cup (V \times W) \cup (U \times W)$.

23. Prove or disprove: There is a constant k_0 such that every k_0 -connected graph is Hamiltonian.

24. A football is made of pentagons and hexagons, not necessarily of regular shape. They are sewn together so that their seams form a cubic graph. How many pentagons does the football have?

25. Does every minimal non-planar graph G (i.e. every non-planar graph G whose proper subgraphs are all planar) contain an edge e such that $G - e$ is maximally planar? Does the answer change if we define “minimal” with respect to minors rather than subgraphs?

26. Show that adding a new edge to a maximal planar graph of order at least 6 always produces both a subdivision of K_5 and a subdivision of $K_{3,3}$ as a subgraph.

27. Show that a 2-connected plane graph is bipartite if and only if every face (i.e. region) is bounded by an even cycle.

28. (a) What does the plane dual of a plane tree look like?

(b) Show that the plane dual of a plane multigraph is connected.

29. Let G and G^* be dual plane graphs. Prove the following statements:

(a) If G is 2-connected with at least three faces, then G^* is 2-connected.

(b) If G is 3-connected, then G^* is 3-connected.

(c) If G is 4-connected, then G^* does not need to be 4-connected.

¹The t_0 -Tough Conjecture: There is a constant t_0 such that every t_0 -tough graph is Hamiltonian. The conjecture was formulated by Chvatal in 1973 and is still open. In the meanwhile it has been shown that $t_0 \geq \frac{9}{4}$ must hold, if the t_0 -Tough Conjecture is true.