Advanced and algorithmic graph theory
Summer term 2020

Second work sheet

13. Prove the following theorem stated in the lecture.

    Let G be a connected graph. Assume that $DFS(s)$, i.e. a depth first search starting at some $s \in V(G)$, has been performed in G. Consider the classification of edges from $E(G)$ into *tree edges* (collected in the set T) and *backward edges* (collected in the set B), as well as their orientation according to $DFSNum$ (cf. lecture). This orientation allows the specification of a starting vertex and an end vertex for every edge (cf. lecture). For all $v \neq s$ and for all $\{v,w\} \in E(G)$ starting at v the following holds: $\{v,w\}$ and the tree edge ending at v belong to the same block if and only if one of the following conditions holds: (a) $\{v,w\}$ is a backward edge or (b) $\{v,w\}$ is a tree edge which is not a leading edge.

14. Prove the following theorem stated in the lecture.

    Let G be a connected graph. Assume that $DFS(s)$, i.e. a depth first search starting at some $s \in V(G)$, has been performed in G. The following statements hold:

    (a) The root s is a cut-vertex if and only if there exists more than one leading edge incident to s.

    (b) A vertex $v \in V(G) \setminus \{s\}$ is a cut-vertex if and only if there exists at least one leading edge starting at v.

15. Construct the block-cut-vertex graph $bc(G)$ of the graph G represented in Figure 1. To this end perform the following steps

    (a) Apply the depth-first-search approach as described in the lecture starting at vertex v_{12}. Determine the depth-first-search tree, the forward edges and the backward edges including their orientations.

    (b) Determine the values of $DFSNum(v)$ and $LowPoint(v)$, for all vertices v, as well as the leading edges.

    (c) Determine the blocks and the cut-vertices and construct the $bc(G)$

16. Consider the following theorem of Chvátal and Erdös (cf. lecture)

    > Let G be a graph with at least 3 vertices ($|V(G)| \geq 3$), connectivity number $\kappa(G)$ and stability number $\alpha(G)$. If $\kappa(G) \geq \alpha(G)$, then G is Hamiltonian.

    Show that this theorem is best possible, in the sense that there exist non-Hamiltonian graphs G with $\alpha(G) = \kappa(G) + 1$. Illustrate this fact for the Petersen graph and the complete bipartite graph $K_{r,r+1}$, $r \in \mathbb{N}$.

17. Consider the following theorem of Ore, Bermond and Linial (cf. lecture)

    > Let G be a 2-connected graph in which $d(x) + d(y) \geq d$ holds for any two non-adjacent vertices $x, y \in V(G)$ and some arbitrary but fixed natural number d. Then there exists a cycle of length at least $\min\{n, d\}$ in G.

    Show that this theorem directly implies the following two results

    (1) If in a graph G with $|V(G)| \geq 3$, the inequality $d(x) + d(y) \geq n$ holds for any two vertices $x, y \in V(G)$ such that $\{x,y\} \notin E(G)$, then G is Hamiltonian. (Due to Ore, cf. lecture.)

    (2) A graph G with $n := |V(G)| \geq 3$ and minimum degree $\delta(G) \geq n/2$ is Hamiltonian. (Due to Dirac, cf. lecture.)

18. Let G be a connected graph with $d := \min\{d(x) + d(y) \colon x, y \in V(G), \{x, y\} \notin E(G)\}$. If $d \geq 2\delta(G)$ holds, then G contains a path with at least $\min\{d + 1, n\}$ vertices.

19. Construct

    (a) a non-Hamiltonian connected 4-regular graph with 11 vertices, and

    (b) a non-Hamiltonian 2-connected 4-regular graph.

20. Show that the d-dimensional cube Q_d is Hamiltonian (cf. Exercise No. 2 for the definition of Q_d).

21. A graph G is called *Hamiltonian connected* iff for any two different vertices $u, v \in V(G)$, $u \neq v$, there exists a Hamiltonian u-v-path in G. Prove the following statements:

    (a) If $d := \min\{d(x) + d(y) \colon x, y \in V(G), \{x, y\} \notin E(G)\} \geq |V(G)| + 1$ holds, than also $\kappa(G) \geq \alpha(G) + 1$ holds.

    (b) If $\kappa(G) \geq \alpha(G) + 1$ holds for some graph G, then G is Hamiltonian connected.

    (c) If the $(n + 1)$-st Hamiltonian hull $\mathcal{H}_{n+1}(G)$ of G is isomorphic to K_n, then G is Hamiltonian connected.

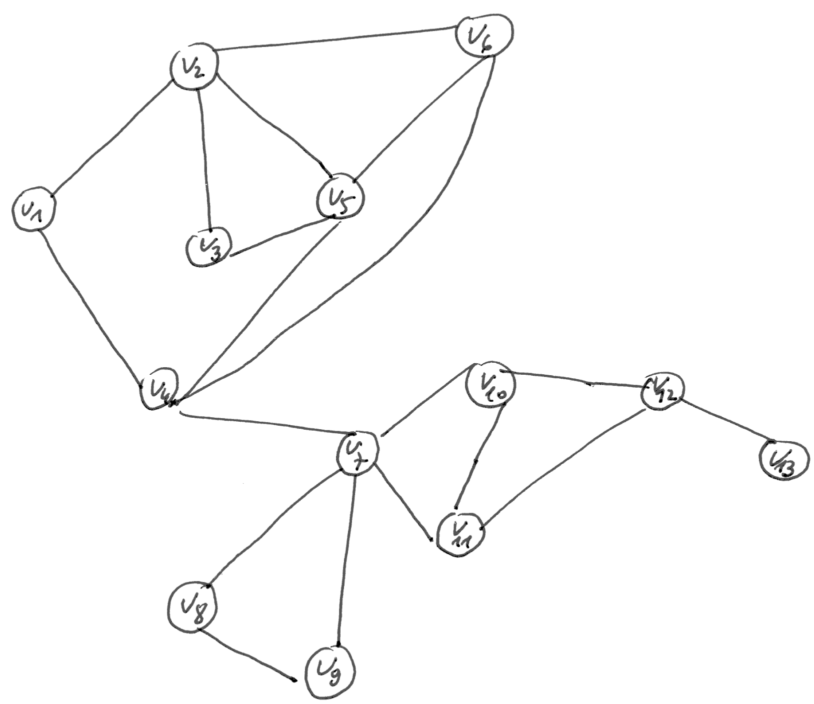

Figure 1: Graph for Exercise 15