Advanced and algorithmic graph theory
Summer term 2020

First work sheet

1. Show that $rad(G) \leq diam(G) \leq 2rad(G)$ hold for every graph G, where $rad(G)$ denotes the radius of graph G and $diam(G)$ denotes its diameter as defined in the lecture.

2. Let $d \in \mathbb{N}$ and $V = \{0,1\}^d$, thus V is the set of all 0-1-sequences of length d. The graph with vertex set V in which two such sequences form an edge iff they differ in exactly one position, is called the d-dimensional cube and is denoted by Q_d. Determine the average degree, the number of edges, the diameter, the girth and the circumference of Q_d.

   (Hint for the circumference: induction on d.)

3. Prove that a graph G with $rad(G) \leq k$ and $\Delta(G) \leq d$, for some $k, d \in \mathbb{N}$, $d \geq 3$, has less than $\frac{d}{d-2}(d-1)^k$ vertices.

   Hint: Consider a central vertex z and the sets D_i of vertices at distance i from z. Estimate the cardinality of D_i, for $i \in \{0, 1, \ldots, k\}$.

4. Prove that a graph G with minimum degree $\delta := \delta(G)$ and girth $g := g(G)$ has at least $n_0(\delta, g)$ vertices[1], where

$$n_0(\delta, g) := \begin{cases} 1 + \delta \sum_{i=0}^{r-1} (\delta - 1)^i & \text{if } g =: 2r+1 \text{ is odd} \\ 2 \sum_{i=0}^{r-1} (\delta - 1)^i & \text{if } g =: 2r \text{ is even} \end{cases}$$

5. Determine the connectivity $\kappa(G)$ and the edge connectivity $\lambda(G)$ for

   (a) $G = P_m$ being a path of length m,

   (b) $G = C_n$ being a cycle of length n,

   (c) $G = K_n$ being a complete graph with n vertices,

   (d) $G = K_{m,n}$ being a complete bipartite graph with m and n vertices in its partition sets, respectively, i.e $K_{m,n} := (A \cup B, E)$ with $|A| = m$, $|B| = n$ and $E = \{(a, b) \colon a \in A, b \in B\}$,

   (e) G being the d dimensional cube.

6. Prove the following theorem of Dirac (1960): Any k vertices of a k-connected graph, $k \geq 2$, lie on a common cycle.

7. Let G be a $2k$-edge connected graph for some $k \in \mathbb{N}$. Show that G contains at least k edge-disjoint spanning trees. Is this result best possible, i.e. is there any $2k$-edge connected graph, which does not contain $k+1$ edge-disjoint spanning trees, for some $k \in \mathbb{N}$? Given an arbitrary $k \in \mathbb{N}$, can you find a $2k$-edge connected graph, which does not contain $k+1$ edge-disjoint spanning trees?

8. Let $G = (V, E)$ be a graph and let T be a normal (rooted) tree with root r in G Show that the following holds for any normal tree T in G.

   (a) Any two vertices $x, y \in V(T)$ are separated in G by the set $\lceil x \rceil \cap \lceil y \rceil$.

   (b) If $S \subseteq V(T) = V(G)$ and S is down-closed (i.e. S contains the down-closure of any element $s \in S$), then the components of $G - S$ are spanned by the sets $\lfloor x \rfloor$ with x minimal in $V(T) - S$.

[1] Interestingly, one can obtain the same bound by replacing $\delta(G)$ by $d(G)$. More precisely, if $d(G) \geq d \geq 2$ and $g(G) \geq g$, for some $g \in \mathbb{N}$, then $|G| \geq n_0(g, d)$ holds, where $n_0(g, d)$ is defined as in Exercise 4. This was proved in N. Alon, S. Hoory and N. Lineal, The Moore bound for irregular graphs, *Graphs and Combinatorics* **18**, *2002, 53–57.*

9. [2] Let G be a connected graph and let $r \in V(G)$. Show that there exists a normal spanning tree T rooted at r in G.

10. A graph G is called *cubic*, if all vertices of G have degree 3, i.e. $d_G(v) = 3$, for all $v \in V(G)$. Show that for a cubic graph G the equality $\lambda(G) = \kappa(G)$ holds, i.e. the vertex connectivity equals the edge connectivity.

11. (a) Show that for a graph G with $diam(G) = 2$ the equality $\lambda(G) = \delta(G)$ holds.
    (b) Let G be a graph with $|V(G)| \geq 2$ such that $d(u) + d(v) \geq n - 1$ holds, for all $u, v \in V(G)$ with $\{u, v\} \notin E(G)$. Show that $\lambda(G) = \delta(G)$.

12. (a) Show that for the d-dimensional cube Q_d, $d \in \mathbb{N}$, $d \geq 2$, the equality $\kappa(Q_d) = \delta(Q_d) = d$ holds. (See Exercise No. 2 for the definition of Q_d.)
    (b) A *Halin graph* H is defined as a graph obtained from a tree T without vertices of degree 2 by adding to it a cycle which joins all the leaves of T. Show that $\kappa(H) = \delta(H) = 3$ holds for any Halin graph H.

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[2] One possibilty to solve this exercise (probably not the simplest one) is to show that the edges traversed according to following precedure P form a normal spanning tree with root r in a connected graph G.

P: Starting from r move alonge the edges of G, going whenever possible to a vertex not visited so far. If there is no such a vertex, go back along the edge by which the current vertex was first reached, unless the current vertex is r, in which case the procedure terminates.

Normal trees generate by the procedure P above are called *depth first search trees*.