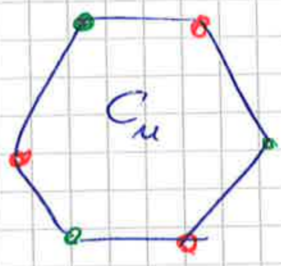
In all other cases the bound can be improved

Theorem (Brooks 1941)

Let G be a connected graph. If G is neither complete nor an odd cycle, then $\chi(G) \leq \Delta(G)$.

Proof: By induction on $|G|$. If $\Delta(G) = 2$ then $|G|$ is a path P_n or a cycle C_n with an even number of vertices. Then $\chi(P_n) = 2 \leq \Delta(P_n)$ and $\chi(C_n) = 2 \leq \Delta(P_n)$

P_n



(If $\Delta(G) = 1$ then G consists of a single edge thus $G = K_2$ so this case is excluded in Brook's theorem)

So we can assume w.l.o.g. $\Delta := \Delta(G) \geq 3$.

In this case $|G| \geq 4$. Assume $|G| = 4$ (as induction basis).

A connected G with $|G| = 4$ and $\Delta(G) \geq 3$ is one of the following graphs (up to isomorphism)

$G_4 = K_4$ is excluded. Moreover $\chi(G_1) = 2 \leq \Delta(G_1) = 3$, $\chi(G_i) = 3 \leq \Delta(G_i)$ for $i \in \{2,3\}$.

So the induction basis holds.

Induction step: Assume $\chi(G') \leq \Delta(G')$ holds for all G' with $|G'| < |G|$, such that G' is neither a complete graph nor an odd cycle. We show that the statement also holds for G.

Assume by contradiction that $\chi(G) > \Delta(G) =: \Delta$.

Brooks 1a

Let $v \in V(G)$ and $H := G - v$. Then $\chi(H) \leq \Delta(H) \leq \Delta(G) \leq \Delta$

by the induction assumption.

Why can the induction's assumption be applied in this case?

Consider a connec. component H' of H.

If H' is neither a complete graph nor an odd cycle we have $\chi(H') \leq \Delta(H')$.

Otherwise, if H' is a complete graph or an odd cycle

we have $\chi(H') = \Delta(H') + 1 \overset{!}{\leq} \Delta$, where the

> check it out

last inequality holds because $\deg(v) = \Delta(H'), \forall v \in V(H')$

in this case and $\exists v' \in V(H')$ such that $\{v', v\} \in E(G)$

(because G is connected!

Thus $\Delta = \Delta(G) \geq \deg_G(v') = \Delta(H') + 1$.

Thus H can be Δ-colored by G cannot

Consider Δ-coloring of G. We show that they have some properties:

(*) Every Δ-coloring of H uses all colors $1, 2, \dots \Delta$ on the neighbors of v, in particular $deg_G(v) = \Delta$ and $\forall$ color there exist a neighbor of v colored by that color

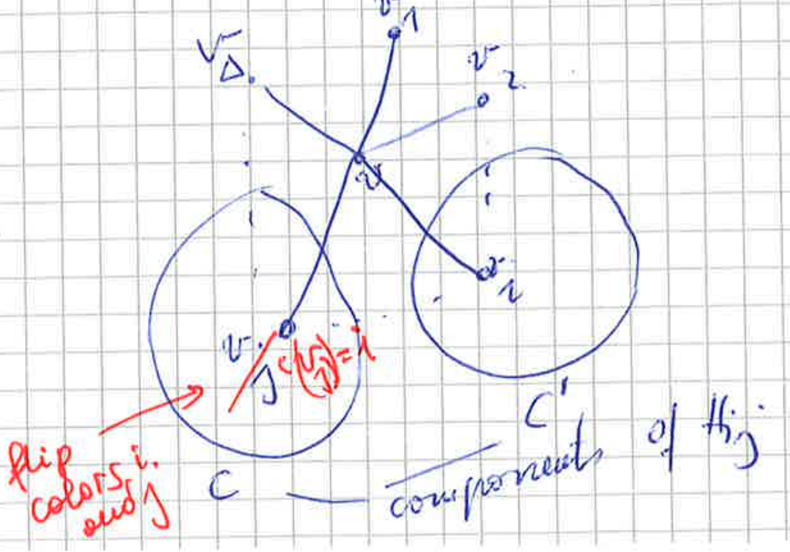
Given a Δ-coloring of H, let us denote by v_i the neighbor of v colored by i.

Further $\forall$ (i,j) with $i \neq j$ $i,j \in \{1, 2, \dots, \Delta\}$ let H_{ij} be the subgraph of H induced by the vertices of H colored by i or j.

The following properties hold

(**) $\forall$ $i \neq j$ the vertices v_i and v_j lie in a common component C_{ij} of H_{ij}

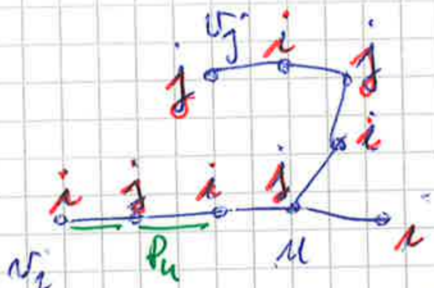
otherwise interchange colors i and j in one of the components containing v_i or v_j; then v_i and v_j would hold the same color contradicting (*)



Brooks 3

(***) C_{ij} is a v_i-v_j-path.

Indeed let P be a v_i-v_j-path in C_{ij}.
Since $d_H(v_i) \le \Delta - 1$ (v_i is a neighbor of $v \notin V(H)$) the neighbors of v_i in H have pairwise different colors otherwise we could recolor v_i in contradiction to (*). Hence $\deg_{C_{ij}}(v_i) = 1$ and analogously $\deg_{C_{ij}}(v_j) = 1$.
Thus the neighbor of v_i in P is the only neighbor in C_{ij}, and a similar statement holds for v_j.
Thus if $C_{ij} \neq P$, then P has ~~an~~ inner vertex with 3 identically colored neighbors in H. Let u be the first such vertex in P.



So at most $\Delta-2$ colors are used on the neighbors of u ($\deg_H(u) \le \Delta$).

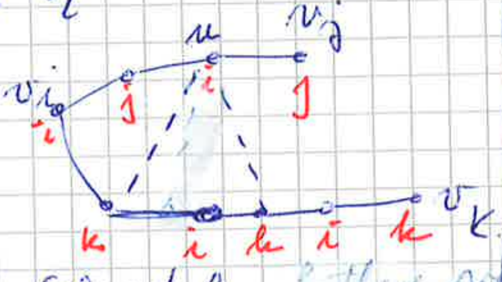
So we may recolor u.

(colors in red) But then P_u is a component of C_{ij} ~~and~~ this comp. contains v_i but not v_j contradicting (**).

We show the next property:

(****) $\forall$ pairwise distinct i,j,k the paths C_{ij} and C_{ik} meet only in v_i.

Otherwise a vertex u would exist with 2 neighbors colored j and 2 neighbors colored k.



So there are $\Delta-2$ colors used in neighbors of u; so recolor u and then v_i, v_j lie in ~~$\Delta-2$ colors used~~ different ~~comp. contradicting~~

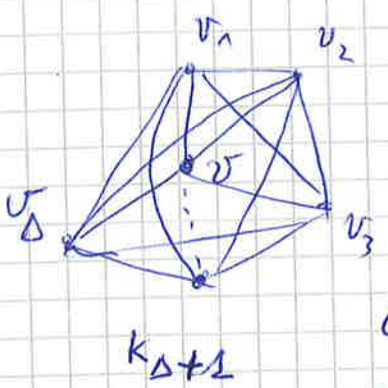
components of H_{ij}, and this contradicts (**).

The rest of the proof goes as follows:

- If the neighbors of v are pairwise adjacent then each of them has Δ neighbors in $N(v) \cup \{v\}$. So G' ~~as~~ follows

$$G' = G[N(v) \cup \{v\}] = K_{\Delta+1} \text{ is a complete graph}$$

on $\Delta+1$ vertices. Now observe that $G' = G$ must hold because if there is a $u \notin N(v) \cup \{v\}$

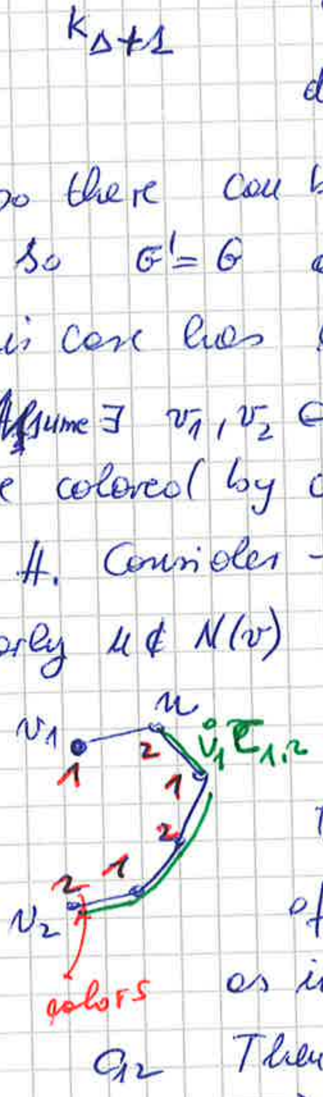


$u \in V(G)$ then u must be connected to some vertex in $N(v) \cup \{v\}$ (because G is connected). But we have already $\deg_{G'}(x) = \Delta \;\; \forall x \in N(v) \cup \{v\}$ and $\deg_G(x) \le \Delta$

so there can be no edge $\{x, u\} \in E(G)$.

So $G' = G$ and G is a complete graph $K_{\Delta+1}$, but this case has been excluded.

- Assume $\exists\, v_1, v_2 \in N(v)$ with $\{v_1, v_2\} \notin E(G)$ and v_1, v_2 are colored by colors 1 and 2, resp., in some Δ-coloring of H. Consider the path $C_{1,2}$ and let u be the neighbor of v_1 in $C_{1,2}$. Clearly $u \notin N(v)$ (otherwise $u = v_2$) and $c(u) = 2$.



Consider $C_{1,3}$ for some color $3 \notin \{1, 2\}$ and interchange colors 1 and 3 in $C_{1,3}$. Denote by c' the resulting feasible Δ-coloring of H and let v_1', H_{ij}', C_{ij}' etc be as introduced above but for the c' coloring.

Then $v_3' = v_1$ and u is in $C_{2,3}'$ (because $c'(u) = c(u) = 2$). ~~By (****)~~ Since the subpath $\mathring{v}_1 C_{12}$ (see the picture) retains its original coloring, we have $\mathring{v}_1 C_{12} \subseteq C_{12}'$. So $u \in C_{23}' \cap C_{12}^{\circ}$ and $u \notin N(v)$. ~~thus u is an inner vertex of C_{23}', C_{12}', contradicting (****)~~

Thus u is an inner vertex,
a common inner vertex of $C'_{1,2}$ and $C'_{2,3}$ and
this contradicts (****).

Thus assuming $\chi(G) > \Delta(G)$ yields to a contradiction (the neighbors of v can neither be pairwise connected by edges, nor can there exist a disconnected pair!).

Thus this assumption is wrong, the opposite $\chi(G) \leq \Delta(G)$ holds and this completes the proof.