Chapter 8: Well-quasi-ordering and tree-decompositions

Motivation: Trees have very nice combinatorial properties, many hard problems are efficiently solvable if the input graph is a tree.

Idea: Detect tree-like structures in general graphs, define an appropriate similarity measure of a graph to a tree. Exploit the similarity to extend the nice combinatorial properties and behaviour of trees to general graphs.

Definition 1

Let X be a ground set and $\preceq$ be a binary relation on X. $\preceq$ is called a **quasi-ordering** *if it is reflexive and transitive. A* **quasi-ordering** *$\preceq$ on X is called a* **well-quasi-ordering (WQO)** *and* **the elements of X are well-quasi-ordered by** *$\preceq$ if for every infinite sequernce x_0, x_1, ..., x_n, ..., there exist two indices i, j, with $i < j$ such that $x_i \preceq x_j$. Such a pair (x_i, x_j) is called a* **good pair** *of the sequence. A sequence containing a good pair is a* **good sequence***. Thus a quasi-ordering on X is a WQO iff every infinite sequence is good. An infinite sequence is* **bad** *if it is not good.*

Chapter 8: Properties of WQOs and examples

Proposition 1

A quasi-ordering $\preceq$ on X is a WQO iff X contains neither an infinite antichain (i.e. an infinite subset any two elements of which are not in relation $\preceq$) nor an infinite strictly decreasing sequence (i.e. a sequence fulfilliong $x_0 \succ x_1 \succ \ldots \succ x_n \succ \ldots$).

Corollary 2

If $\preceq$ is a WQO on X then every infinite sequence in X has an infinite increasing subsequence.

Example:

Let $X := \mathbb{Z}$ and $\preceq := \leq$ (the usual "smaller than or equal to" relation between real numbers).

$\leq$ is a quasi-ordering on $\mathbb{Z}$ but not a WQO

$-1, -2, \ldots, -n, \ldots$ is a bad sequence.

Chapter 8: Properties of WQOs (contd)

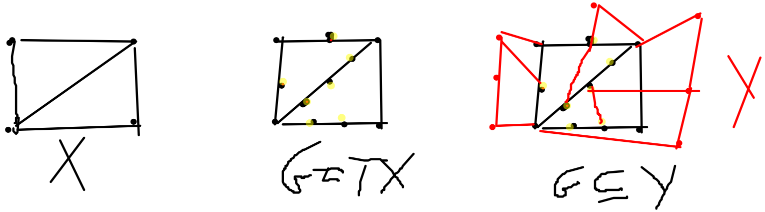
Let $\preceq$ be a WQO on S. For two finite subsets A, B of X, $A, B \subseteq X$, write $A \preceq B$ iff there is an injective mapping $f : A \to B$ such that $a \preceq f(a)$, $\forall a \in A$. This extends $\preceq$ to a WQO on the set $\mathcal{P}_f(X)$ of finite subsets of X.

Lemma 3

If $\preceq$ is a WQO on X, also its above defined extention on $\mathcal{P}_f(X)$ is a WQO.

Chapter 8: Minors and topological minors (revisited)

Recall: A graph X is a topological minor of a graph Y iff Y contains a subdivion TX of X as a subgraph.



Definition 2

$G = IX$ is an **inflation** *of a graph X if G is obtained from X by replacing (i) the vertices x of X by vertex-disjoint connected subgraphs G_x, for all $x \in V(X)$, and (ii) the edges $\{x, y\}$ if X by a nonempty set of $V(G_x)$-$V(G_y)$-edges. Thus G is an IX iff $V(G) = \dot\bigcup_{x \in V(X)} V_x$, where $G[V_x]$ is connected for all $x \in V(X)$ and*
$\{x, y\} \in E(X) \leftrightarrow \exists V_x$-$V_y$-edge in G.
V_x, $x \in X$ are called **the branch sets** *of IX.*

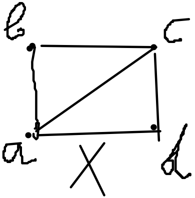
Observe: If $G = IX$ is an inflation of X, then X arises as a contraction of G.

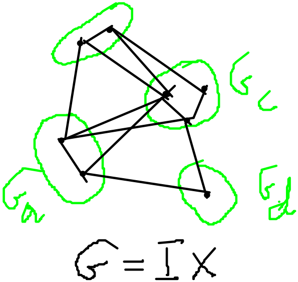
Chapter 8: Minors and topological minors (contd.)

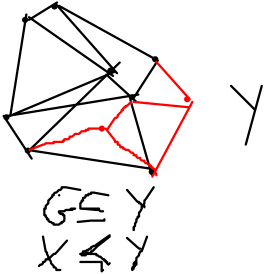
Recall: X is a minor of Y iff Y contrains an inflation IX as a subgraph; notation $X \preceq Y$.

Thus $X \preceq Y \longleftrightarrow \exists \phi \colon Y_1 \subseteq V(Y) \to V(X)$ such that (i) $\forall x \in V(X)$, $Y[\phi^{-1}(x)]$ is connected and (ii) $\forall \{x, x'\} \in E(X)$, $\exists$ a $\phi^{-1}(x)$-$\phi^{-1}(x')$-edge in $E(Y)$.
$\phi^{-1}(x)$ are the branch sets, for $x \in V(X)$,.

If the domain Y_1 of ϕ is the whole $V(Y)$, i.e. $Y_1 = V(Y)$, and $\forall x, x' \in V(X)$, $x \neq x'$, the existence of an $\phi^{-1}(x)$-$\phi^{-1}(x')$-edge in $E(Y)$ implies $\{x, x'\} \in E(X)$, then ϕ is called a **contraction of Y onto X**.







Proposition 4

The minor relation $\preceq$ and the topological minor relation are partial orderings on the class of finite graphs, i.e. both of them are reflexive, antisymmetric and transitive.

If G is an IX and $\exists x \in V(X)$ such $U = V_x$ and $|V_y| = 1$, for all $y \in V(X) \setminus \{x\}$, then me denote X by G/U and v_U for the vertex x of X to which U is contracted. The rest of X can be seen as an induced subgraph of G.

The smallest non trivial case is the contraction of an edge: $U = \{u_1, u_2\}$, $\{u_1, u_2\} \in E(G)$; notation $X = G/e$ (instead of $X = G/U$).

Proposition 5

A finite graph G is an inflation IX of some (finite) graph X iff X can be obtained from G by a sequence of edge contractions, i.e. iff there exist an $n \in \mathbb{N}$, graphs G_0, G_1, ..., G_n and edges $e_i \in G_i$, forall $i \in \overline{0, n-1}$, such that $G_0 = G$, $G_n \simeq X$ and $G_{i+1} = G_i/e_i$, $\forall i \in \overline{0, n-1}$.

Corollary 6

Let X and Y be finite graphs. X is a monot of Y iff there exist an $n \in \mathbb{N}$ and graphs G_0, G_1, ..., G_n such that $G_0 = G$, $G_n \simeq X$ and each G_{i+1} is obtained from G_i by deleting an edge, contracting an edge or deleting a vertex.

Proposition 7

(i) Every subdivion TX of a graph X is also an inflation IX of X, thus every topological minor of a graph is also its ("ordinary") minor.
(ii) If $\Delta(XC) \leq 3$ then every IX contains a TX, thus every minor of maximum degree 3 of a graph is also its topologiocal minor.

Theorem 8

(Kruskal 1960)
The finite trees are well-quasi-ordered by the topological minor relation.

Definition 3

Let G be a graph, T a tree and $\mathcal{V} = (V_t)_{t \in V(T)}$ be a family of sets of vertices $V_t \subseteq V(G)$ indexed by the vertices of T. The pair $(T, \mathcal{V})$ is called a **tree-decomposition of** *G if it satisfies the following three conditions:*

(T_1) $V(G) = \cup_{t \in v(T)} V_t$,

(T_2) $\forall e = \{x, y\} \in E(G)\ \exists t \in V(T)$ *such that* $x \in V_t$ *and* $y \in V_t$,

(T_3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ *whenever* $t_1, t_2, t_3 \in V(T)$ *satisfy* $t_2 \in t_1 T t_3$, *i.e.* t_2 *lies on the unique* t_1-t_3-*path in* T.

$\mathcal{V}$ *and* $G[V_t]$, $t \in V(T)$, *are called the parts of* $(T, \mathcal{V})$.

Lemma 9

(separation lemma)
Let G be a graph and $(T, \mathcal{V})$ be a tree-decomposition of G. Let $\{t_1, t_2\} \in E(T)$ and let T_1, T_2 be the connected components of $T - \{t_1, t_2\}$ wit $t_i \in V(T_i)$, for $i \in \{1, 2\}$. Then $V_{t_1} \cap V_{t_2}$ separates $U_1 := \cup_{t \in V(T_1)} V_t$ from $U_2 := \cup_{t \in V(T_2)} V_t$ in G.

Chapter 8: Tree-decomposition (contd.)

Lemma 10

(tree-decomposition of subgraphs)
Let G be a graph and $(T, \mathcal{V})$ be a tree-decomposition of G. Let $H \subseteq G$ be a subgraph of G. Then the pair $\big(T, (V_t \cap V(H))_{t \in V(T)}\big)$ is a tree decomposition of H.

Lemma 11

(tree-decomposition of contractions)
Let G be a graph and $(T, \mathcal{V})$ be a tree-decomposition of G. Assume that G is an inflation of some graph H with branch sets $U_h \subseteq V(G)$, for $h \in V(H)$. Let $f\colon V(G) \to V(H)$ be the map assigning to each vertex $v \in V(G)$ the index of the branch set containing it, i.e. $v \in U_{f(v)}$ holds for all $v \in V(G)$. $\forall t \in V(T)$ denote $W_t := \{f(v)\colon v \in V_t\}$, and let $\mathcal{W} := (W_t)_{t \in V(T)}$. Then $(T, \mathcal{W})$ is a tree-decomposition of H.

Chapter 8: Tree-decomposition (contd.)

Lemma 12

Let G be a graph and $(T, \mathcal{V})$ be a tree-decomposition of G. Given a set $W \subseteq V(G)$ then either (a) there is a $t \in V(T)$ such that $W \subseteq V_t$ or (b) there exists vertices $w_1, w_2 \in W$ and an edge $\{t_1, t_e\} \in e(T)$ such that w_1 or w_2 lie outside the set $V_{t_1} \cap V_{t_2}$ and are separated by this set in G.

Corollary 13

Let G be a graph and $(T, \mathcal{V})$ be a tree-decomposition of G. Any complete sungraph of G is contained is some part of the tree-decomposition $(T, \mathcal{V})$.

Observation: In a tree-decomposition $(T, \mathcal{V})$ of G the parts *reflect* the structure of the tree T, so G resembles T to the extent that structure of G within each part is negligible. Thus the smaller the parts the closer the resemblance.

Definition 4

Let G be a graph and $(T, \mathcal{V})$ be a tree-decomposition of G. The **width of** $(T, \mathcal{V})$ *is given as* $\max\{|V_t| - 1 \colon t \in V(T)\}$*. The* **tree-width** $tw(G)$ **of** G *is the smallest width of any tree-decomposition of G.*

Remarks:

(1) Every graph has a trivial tree-decomposition $(T, \mathcal{V})$ where T is a singleton with $V(T) = \{x\}$ and $V_x = V(G)$. Thus the tree-width of a graph is well defined.

(2) The tree-width of any tree T eqaals 1: $tw(T) = 1$.

(3) By Lemma 10 and Lemma 11 the tree-width of a graph can never be increased by deletions of edges and/or vertices or contractions.

Proposition 14

If $H \preceq G$, then $tw(H) \leq tw(G)$, where $\preceq$ is the minor relation.

Chapter 8: Tree-width (contd.)

Theorem 15

(Robertson and Seymour 1990)
For every natural number k the graphs of tree-width smaller than or equal to k are well-quasi-order by the minor relation.

Proposition 16

A graph G is chordal iff it has a tree-decomposition into complete parts.

Corollary 17

For any graph G the following holds:

$$tw(G) = \min\{\omega(H) - 1 \colon G \subseteq G,\ G \textit{ is chordal}\}\,.$$

Chapter 8: Tree-width and forbidden minors

Proposition 18

A graph has tree-width less than 3 iff it does not have K_4 as a minor.

Theorem 19

(Robertson and Seymour 1986)
Consider an arbitrary (but fixed) graph H and the class $\mathcal{C}_H$ of all graphs which do not have H as a minor. The tree-width is bounded over $\mathcal{C}_H$ iff H is planar.

Theorem 20

(Robertson and Seymour 1986) For every natural number r, there exists a natural number k such that every graph of tree-width at least k has an $r \times r$ grid as a minor.

Chapter 8: Facts on tree-decomposition and tree-width

Definition 5

An equivalent defintion of tree-decomposition
Given a graph $G = (V, E)$*, a tree-decomposition of* G *is a pair* $(T, \mathcal{V})$ *where* $\mathcal{V} = (V_t)_{t \in V(T)}$ *is a family of sets of vertices* $V_t \subseteq V(G)$ *indexed by the vertices of* T *and*

(T_1) $V(G) = \cup_{t \in v(T)} V_t$,

(T_2) $\forall e = \{x, y\} \in E(G)\ \exists t \in V(T)$ *such that* $x \in V_t$ *and* $y \in V_t$,

(T_3') $\forall v \in V(G)$ *thevertices* $T' := \{t \in V(T) \colon v \in V_t\}$ *build a subtree of* T.

$\mathcal{V}$ *and* $G[V_t]$*,* $t \in V(T)$*, are called the parts of* $(T, \mathcal{V})$*.*
V_t*, for* $t \in V(T)$ *are also called* **bags**.

Observation: The tree-width of a graph is equal to the maximum tree-width of its connected components.

Chapter 8: Facts on tree-decomposition and tree-width (contd.)

Proposition 21

Let G be a graph with $E(G) \neq \emptyset$. Then $tw(G) = 1$ iff G is a forest.

Proposition 22

Every graph G with $tw(G) = k$ has a vertex $v \in V(G)$ with $deg(v) \leq k$

Proposition 23

For a graph G with $|V(G)| = n$ the equality $tw(G) = n - 1$ holds iff G is a clique.

Proposition 24

A graph G has tree-width at most 2 iff G is a subgraph of a series-parallel graph.

See the definition of series-parallel graphs on the next slide.

Chapter 8: Facts on tree-decomposition and tree-width (contd.)

Definition 6

A multigraph is called a **series-parallel graph** *if it is obtained from an independent set by applying the following operations*

(a) *add a new vertex and connect it to an existing vertex by an edge,*

(b) *add a loop,*

(c) *add an edge parallel to an existing edge, or in other words duplicate an existing edge,*

(d) *subdivide an edge by creating a vertex on the edge.*

Proposition 25

It is NP-hard to determnie the tree-width $tw(G)$ of an arbitrary input graph G. There are algorithms which determine whether $tw(G) \leq k$ in time $O(n^k)$, where $|G| = n$ and $k \in \mathbb{N}$ is an arbitrary natural number .

Chapter 8: **Computing tree-decompositions**

Proposition 26

For a graph G with $|G| = n$ and $tw(G) = k$

(i) *a tree-decomposition of width k can be determined in $O(n^{k+\theta(1)})$ time (Arnborg, Corneil, Proskurovski 1987),*

(ii) *a tree-decomposition of width k can be determined in $2^{\tilde{O}(k^3)}n$ time (Bodlaender 1996),*

(iii) *a tree decomposition of width $5k+4$ can be determined in $2^{O(k)}n$ time (Bodlaender et al. 2013),*

(iv) *a tree decomposition of width $O(k\log(k))$ can be determined in time polynomial in n (Feige, Hajiaghai, Lee 2008),*

Chapter 8: Nice tree-decompositions

Let G be a graph and $(T, \mathcal{V})$ be a tree-decomposition of G. We consider T to be rooted (at some arbitrary vertex $r \in V(T)$).

Notations: Let T_x be the subtree of T rooted at x, for any $x \in V(T)$.
$y \in V(T)$ is called a **child** of $x \in V(T)$ in T if $\{x, y\} \in E(T)$ and x lies in the (unique) r-y-path in G, where r is the root of T.
Let $G_x := \cup_{z \in T_x} V_z$, for any $x \in V(T)$.

Definition 7

Let G be a graph and $(T, \mathcal{V})$ be a tree-decomposition of G with a rooted T. The $(T, \mathcal{V})$ is called a **nice tree-decomposition** *if every vertex $x \in V(T)$ is one of the following 4 types:*

- **Leaf:** *x is a leaf in T and $|V_x| = 1$.*
- **Introduce:** *x has one child y in T and $V_x = V_y \cup \{v\}$ for some $v \in V(G)$.*
- **Forget:** *x has one child y in T and $V_x = V_y \setminus \{v\}$ for some $v \in V(G)$.*
- **Join:** *x has two children y_1, y_2 in T with $V_x = V_{y_1} = V_{y_2}$.*

Chapter 8: Nice tree-decompositions and the max independent set problem

Proposition 27

Let G be a graph and $(T, \mathcal{V})$ be a tree-decomposition of G of width w and $O(n)$ vertices where $n := |V(G)|$. $(T, \mathcal{V})$ can be turned into a nice tree decomposition of width w and $O(n)$ vertices in $O(n)$ time.

See N. Betzler, R. Niedermayer and J. Uhlmann, Tree decompositions of graphs: saving memory in dynamic programming, *Discrete Optimization* 3(3), 2006, 220–229.

Proposition 28

Let G be a graph and $(T, \mathcal{V})$ be a nice tree-decomposition of G of width w with $V(T) = O(n)$ and $n := |G|$. The maximum weighted independent set problem in G, i.e. finding an independent set of maximum weight in G for a given vertex weight function $f : V(G) \to \mathbb{R}_+$, can be solved in $O(2^w n)$ time by dynamic programming.