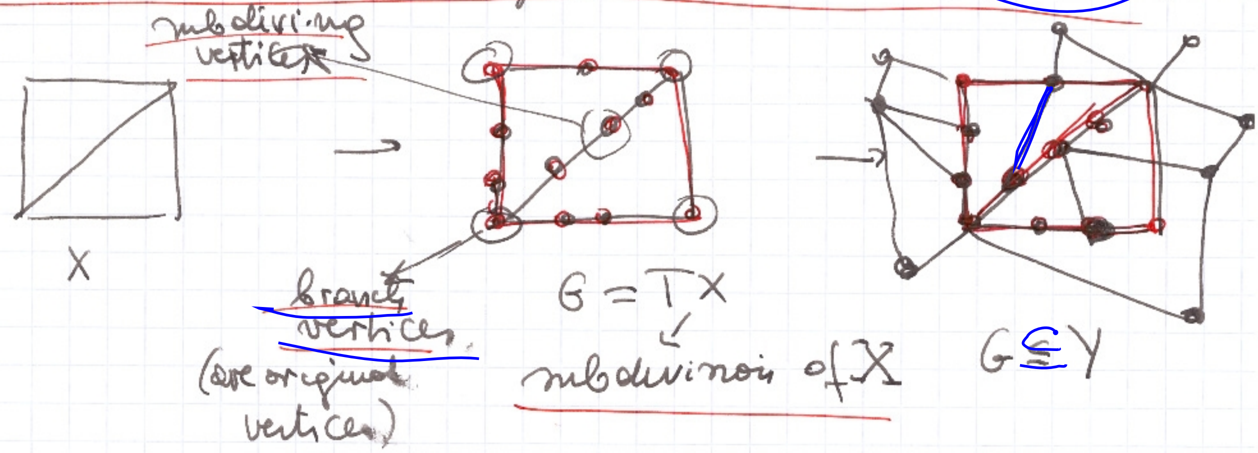
Let $\leq$ be a well quasi ordering on X. For finite subsets $A, B \subseteq X$ write $A \leq B$ if there is an injective mapping $f: A \to B$ s.th. $a \leq f(a)$, $\forall a \in A$. This extends $\leq$ to a quasi ordering in the set of finite subsets of X denoted by $\mathcal{P}_f(X)$.

Lemma 8.3 If X is well-quasi-ordered by $\leq$, then so is $\mathcal{P}_f(X)$.

no proof

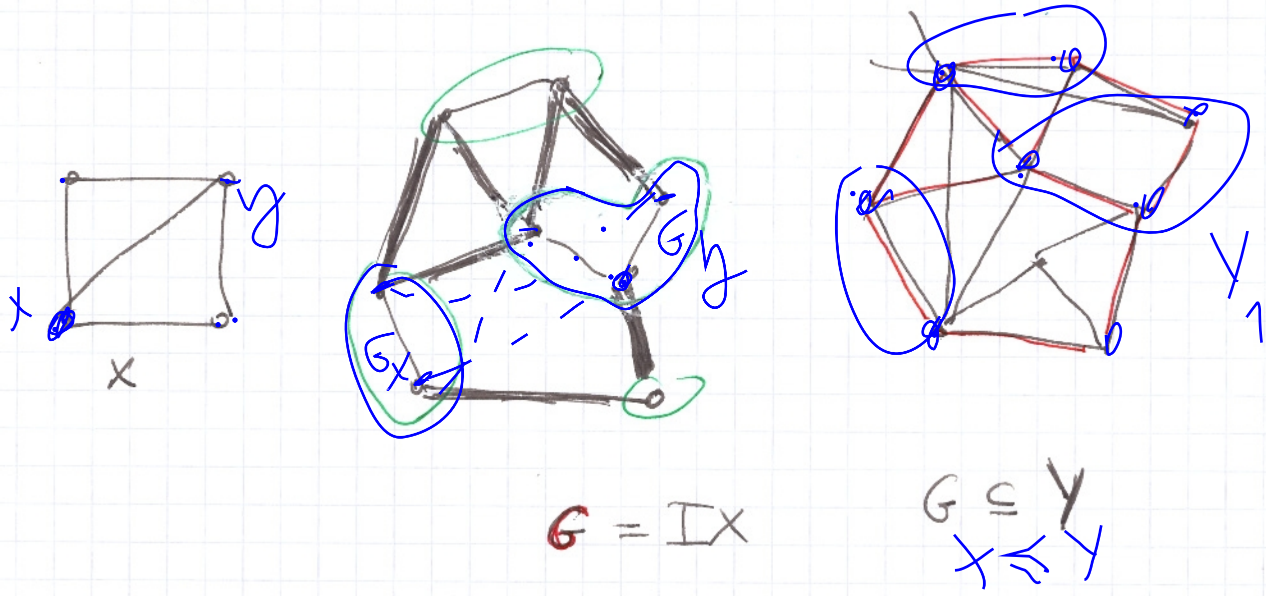
8.3 Minors and topological minors revisited.

TOPOLOGICAL MINORS



X is a topological minor of Y $\overset{Def}{\Longleftrightarrow}$ Y contains a subdivision TX as a subgraph

MINOR



$G = IX$, an inflation of X is obtained from X by replacing the vertices of X by disjoint connected subgraphs G_X and the edges $\{x,y\}$ of X with non-empty sets of $V(G_x)-V(G_y)$ - edges..

In other words: G is an IX if $V(G) = \dot{\bigcup}_{x \in V(X)} V_x$

where $G[V_x]$ is connected $\forall x \in X$ and

$\{x,y\} \in E(X) \iff \exists$ $V_x - V_y$ - edge in G.

MINOR

V_x are called the branch sets of IX.

When G is an inflation of X (IX), then X arises as a contraction of G.

If Y contains an IX as a subgraph then X is a minor of Y, notation $X \preccurlyeq Y$

Thus $X \preccurlyeq Y \iff \exists \varphi: Y_1 \subseteq V(Y) \to V(X)$ such that

(•) $\forall x \in X$ $Y[\varphi^{-1}(x)]$ is connected (in Y)

and (••) $\forall \{x,x'\} \in E(X)$, $\exists$ $\varphi^{-1}(x)-\varphi^{-1}(x')$-edge in $E(Y)$.

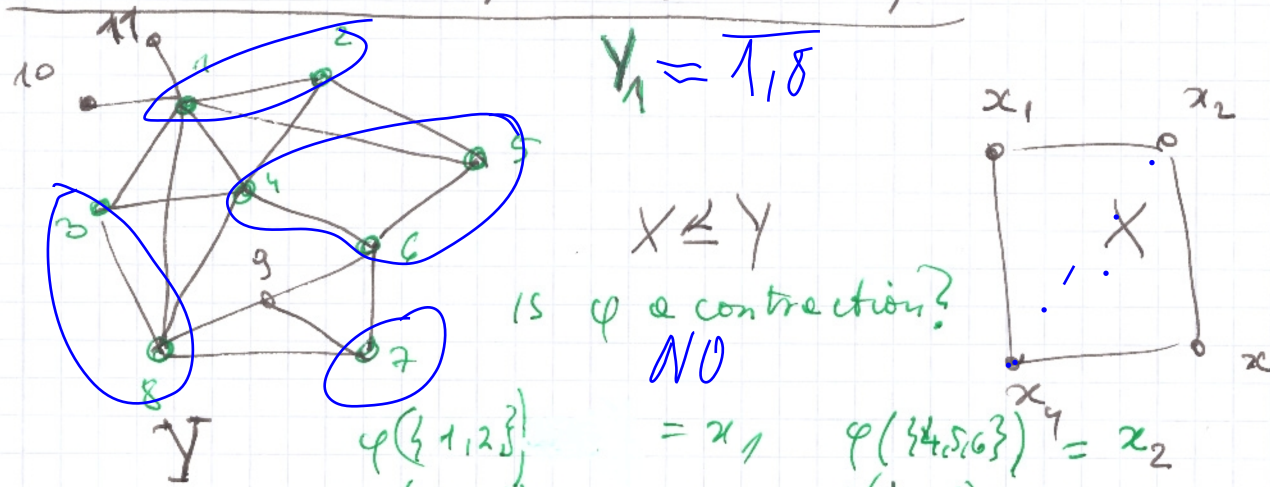
$\varphi^{-1}(x)$ are the branch sets, $\forall x \in V(X)$

If the domain Y_1 of φ is the whole $V(Y)$, i.e. $Y_1 = V(Y)$

and $\forall x, x' \in V(X)$, $x \neq x'$, the existence of an $\varphi^{-1}(x)-\varphi^{-1}(x')$ - edge in Y implies $\{x,x'\} \in E(X)$

we call φ a contraction of Y onto X

Illustration in the previous example

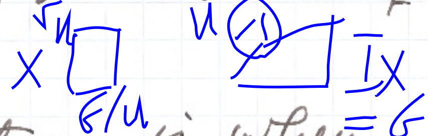


$\varphi(\{1,2\}) = x_1$ $\varphi(\{4,5,6\}) = x_2$

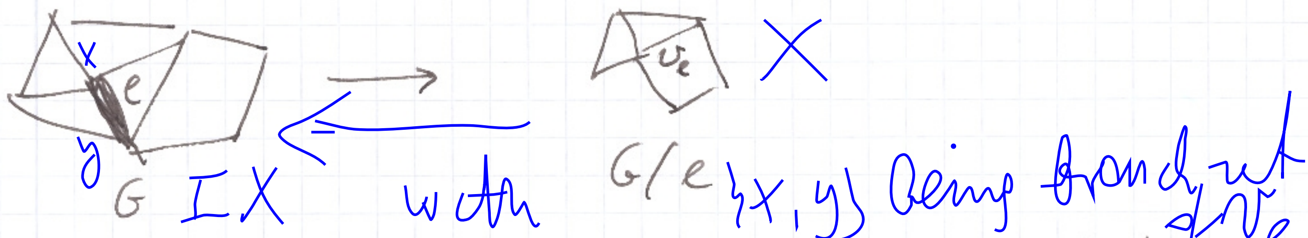
$\varphi(\{3,8\}) = x_4$ $\varphi(\{7\}) = X_3$

Proposition 8.4 The minor relation $\preccurlyeq$ and the topological minor relation are partial orderings in the class of finite graphs, i.e. they are reflexive, antisymmetric and transitive.

If G is an IX with a branch set $U = V_x$ where $x \in X$ and every other branch set containing just a single vertex we also write G/U for the graph X and v_U for the vertex of X to which U contracts, we then think of the rest of X as an induced subgraph of G.



The "smallest" non-trivial of this situation is when $|U| = 2$ and the 2 vertices in U are connected by an edge e. Then we write $X = G/e$ and say that X arises from G by contracting edge e.



Proposition 8.5 A finite graph G is an IX iff X can be obtained from G by a sequence of edge contractions, i.e. iff $\exists$ graphs $G_0, \dots, G_n$ and edges $e_i \in G_i$ such that $G_0 = G$, $G_n \simeq X$ and $G_{i+1} = G_i / e_i$ $\forall\, 0 \le i < n$.

Corollary 8.6 Let X and Y be finite graphs. X is a minor of Y iff there are graphs $G_0, \dots, G_n$ such that $G_0 = Y$, $G_n \simeq X$ and each G_{i+1} arises from G_i by deleting an edge, contracting an edge or deleting a vertex.

And then the relationship between minors and topological minors:

Proposition 8.7

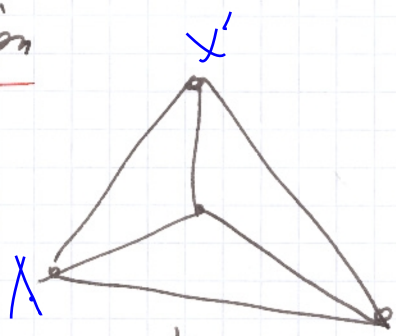
(i) Every TX is also an IX, thus every topological minor of a graph is also its (ordinary) minor

(ii) If $\Delta(X) \leq 3$, then every IX contains a TX, thus every minor with maximum degree at most 3 of a graph is also its topological minor.

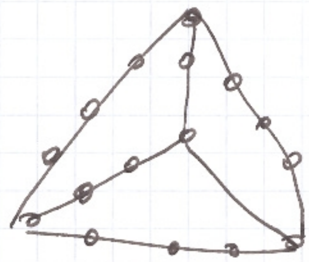
top. minor $\Rightarrow$ minor
$\not\Leftarrow$ except for $\Delta(X) \leq 3$

(no proof)

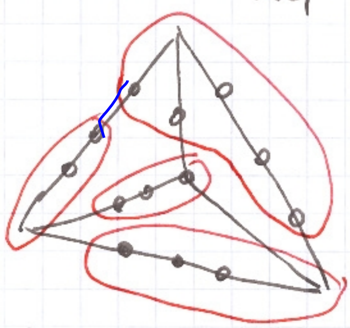
Illustration



K_4



A subdivision of K_4 (TK_4)



Homework: Find X with $\Delta(X) > 3$ and Y s.th. $X \preccurlyeq Y$ but X is not a top. minor of Y

TK_4 considered as an IK_4

Now let us consider the embedding of a graph into another

Def: An embedding of G in H is an injective map $\varphi: V(G) \to V(H)$ such that the kind of structure we are interested in is preserved.

Thus (•) "φ embeds G in H as a subgraph" if it preserves the adjacency of vertices $\{x, y\} \in E(G) \Rightarrow \{\varphi(x), \varphi(y)\} \in E(H)$ (*)

(••) "φ embeds G in H as an induced subgraph"

* and $\{\varphi(x), \varphi(y)\} \in E(H) \Rightarrow \{x,y\} \in E(G)$

if it preserves both adjacency and non adjacency.

(•••) $\varphi: V(G) \cup E(G) \to V(H) \cup P(H)$

where $P(G)$ is the set of paths in H, then "φ embeds G in H as a topological minor" if

$\forall v \in V(G)$ $\varphi(v) \in V(H)$ and $\forall e \in E(G)$ $\varphi(e) \in P(H)$ and different edges $e, e' \in E(G)$, $e \neq e'$ are mapped on paths $\varphi(e), \varphi(e')$ which do not share inner vertices.

(••••) A $\varphi: V(G) \to 2^{V(H)}$ (the set of subsets of H without the proper subset $\emptyset$)

embeds G in H as a minor if $\forall v \in V(G)$

$H[\varphi(v)]$ is a connected subgraph of H (single vertices are consider as connected)

$\forall v, v' \in G$, $v \neq v' \Rightarrow \varphi(v) \cap \varphi(v') = \emptyset$, and

$\{v, v'\} \in E(G) \Rightarrow \exists$ $\varphi(v)$-$\varphi(v')$-path in H

Homework: Define an embedding of G in H as

(a) a spanning subgraph

(b) as an induced minor

8.4 The graph minor theorems for trees

Theorem 8.8 (Kruskal 1960)

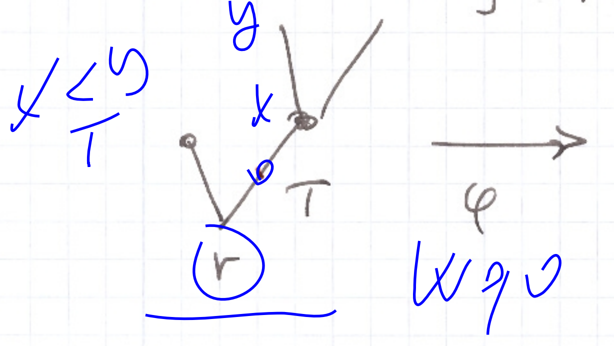
The finite trees are well quasi ordered by the topological minor relation.

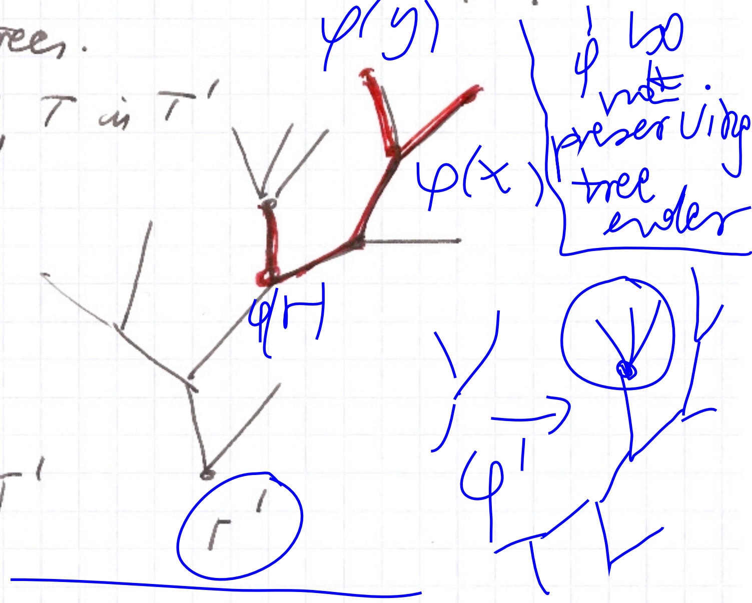
(no proof) but a remark:

~~The proof~~ The proof is based on an embedding between rooted trees which strengthens the usual embedding as a topological minor. Consider 2 trees T and T' with roots r and r', respectively. We write $T \leq T'$ if there is an isomorphism φ from some subdivision of T to a subtree T'' of T' that preserves the tree order in $V(T)$ associated with T and r, i.e. if $x < y$ in T, then $\varphi(x) < \varphi(y)$ in T'.

It can be easily checked that this is a ~~[illegible]~~ qo in the set of finite rooted trees.

Example: an embedding φ of T in T' showing $T \leq T'$





8.5 Tree decompositions

Def: Let G be a graph, T a tree and $\mathcal{V} = (V_t)_{t \in V(T)}$ be a family of vertex sets $V_t \subseteq V(G)$ induced by the vertices t of T. The pair $(T, \mathcal{V})$ is called a tree-decomposition of G if it satisfies the following three conditions:

(T_1) $\quad V(G) = \bigcup_{t \in T} V_t$

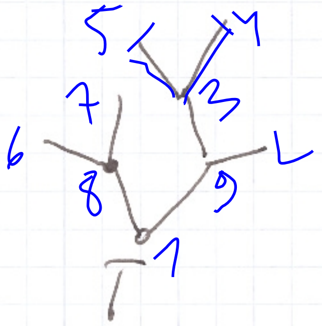
(T_2) $\quad \forall e \in E(G)\ \exists\ t \in T$ such that both endpoints of e lie in V_t.

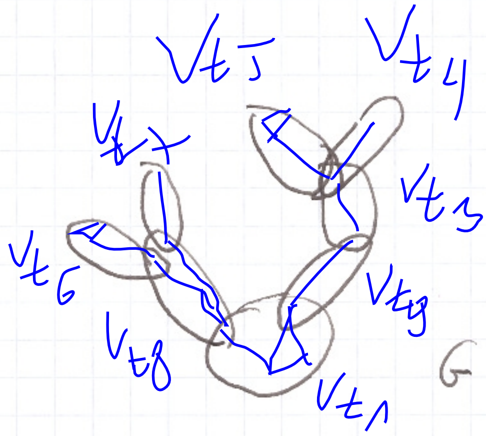
(T_3) $\quad V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever $t_1, t_2, t_3 \in T$ satisfy $t_2 \in t_1 T t_3$, i.e. t_2 lies on the $t_1 - t_3$-path in T.

t_1 ——·——x—— t_2 —— t_3

$\mathcal{V}$ and $G[V_t]$, $\forall t \in T$, are called the parts of $(T, \mathcal{V})$

Example





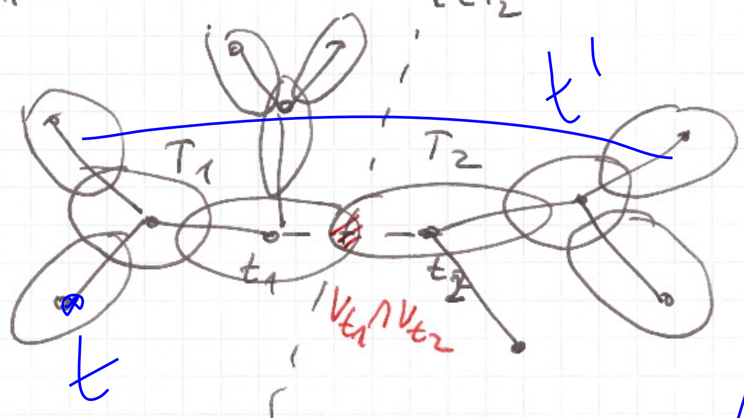
$V_{t_2} = \emptyset$

One of the most important properties of tree-decompositions: it transfers the separation properties of its tree to the graph decomposed

Lemma 8.9 (separation lemma)

Let $\{t_1 t_2\} \in E(T)$ and let T_1, T_2 be the components of $T - \{t_1 t_2\}$ with $t_i \in T_i$, $i \in 1,2$. Then $V_{t_1} \cap V_{t_2}$ separates $U_1 := \bigcup_{t \in T_1} V_t$ from $U_2 := \bigcup_{t \in T_2} V_t$ in G.

Example



$u \in U_1 \cap U_2$
$u \in U_1 \Rightarrow \exists t \in V(T_1)$ $u \in V_t$
$u \in U_2 \Rightarrow \exists t' \in V(T_2)$ $u \in V_{t'}$
(T3) $V_t \cap V_{t'} \subseteq V_{t_i}$, $i \in 1,2$

Proof Both t_1 and t_2 lie on every path $t - t'$ in T with $t \in T_1$, $t' \in T_2$. Therefore $U_1 \cap U_2 \subseteq V_{t_1} \cap V_{t_2}$ by (T3). So we need to show that $\nexists$ edge $\{u_1 u_2\} \in E(G)$ with $u_1 \in U_1 \setminus U_2$ and $u_2 \in U_2 \setminus U_1$ $(U_1 \cup U_2 = V(G))$.
If $\{u_1 u_2\} \in E(G)$ is such an edge, then by T_2 $\exists t \in T$ with $u_1, u_2 \in V_t$. But then by the choice of u_1 and u_2 we have $t \notin T_2$ and $t \notin T_1$ ↯.

$t \in T_2 \Rightarrow u_1 \in U_1 \setminus U_2$ ↯ $t \in T_2$ $u_1 \in \bigcup_{t \in T_2} V_t = U_2$ ↯ □

The next lemma shows that tree decompositions are passed on to subgraphs.

Lemma 8.10 If $H \subseteq G$ the pair $(T, (V_t \cap V(H))_{t \in T})$ is a tree decomposition of H let $(T, \mathcal{V})$ be a t.-d. of G. and similarly for contractions.

Proof: homework

Lemma 8.11 Suppose that G is an IH with branch set U_h, $h \in V(H)$. Let $f: V(G) \to V(H)$ be the map assigning to each vertex of G the index of the branch set containing it. $\forall t \in T$ let $W_t = \{f(v) : v \in V_t\}$ and let $\mathcal{W} = (W_t)_{t \in T}$. Then $(T, \mathcal{W})$ is a tree decomposition of H. Here we assumed $(T, \mathcal{V})$ to be a tree-decomp. of G.

Proof (T_1) and (T_2) for $(T, \mathcal{W})$ follow immediately from (T_1) and (T_2) for $(T, \mathcal{V})$

Now let us show (T_3). Let $t_1, t_2, t_3 \in T$ be such that t_2 lies on the path from t_1 to t_3 in T. We show $W_{t_1} \cap W_{t_3} \subseteq W_{t_2}$

Consider a vertex $h \in W_{t_1} \cap W_{t_3}$ of H and show $h \in W_{t_2}$.

By def. of W_{t_1} and W_{t_3}, $\exists$ vertices $v_1 \in V_{t_1} \cap U_h$ and $v_3 \in V_{t_3} \cap U_h$; indeed

$$h \in W_{t_1} \cap W_{t_3} \Rightarrow h \in \underbrace{\{f(v) : v \in V_{t_1}\}}_{W_{t_1}} \cap \underbrace{\{f(v) : v \in V_{t_3}\}}_{W_{t_2}}$$

$\Rightarrow$ ~~$\exists v_1 \in U$~~ $\exists v_i \in V_{t_i}$, $h = f(v_1)$ $\Leftrightarrow v_1 \in U_h$ and $h = f(v_2)$ $\Leftrightarrow v_2 \in U_h$

$\Rightarrow v_1 \in U_h \cap V_{t_1} \wedge v_2 \in U_h \cap V_{t_2}$

Since U_h is connected in G and V_{t_2} separates ~~then~~ v_1 from v_3 in G by Lemma 8.9, V_{t_2} has a vertex in $U_h \Rightarrow h \in W_{t_2}$. ∎