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3) By lemma 8.10 and lemma 8.11 the tree-width of a graph will never be increased by deletion or contraction.

Proposition 8.14 If $H \preceq G$ then $tw(H) \leq tw(G)$, where $\preceq$ is the minor relation.

Theorem 8.15 (Robertson & Seymour 1990)
$\forall$ integer $k > 0$ the graphs of tree width $< k$ are well-quasi-ordered by the minor relation.
Without a proof (a proof can be read in chapter 12 of R. Diestel, Graph theory, Springer, 2012.
came so far on June 22, 2020

Proposition 8.16 G is chordal iff G has a tree-decomposition into complete parts.

$(T, \mathcal{V} = (V_x)_{x \in V(T)})$
$\forall x \in V(T)$ $G[V_x]$ is complete

Proof Induction on $|G|$.
$\Rightarrow$ Assume that G has a tree-decomposition $(T, \mathcal{V})$ such that $G[V_t]$ is complete $\forall t \in T$. Choose such a $(T, \mathcal{V})$ with $|T|$ minimal. If $|T| = 1$ then $V_t = V(G)$ and hence G is complete $\Rightarrow$ chordal.. So assume w.l.o.g $|T| \geq 2$. Let $\{t_1, t_2\} \in E(T)$; let T_1, T_2 be the trees of the forest $T \setminus \{t_1 t_2\}$ such that $t_i \in V(T_i)$, $i \in \overline{1,2}$, and let $U_i = \bigcup_{t \in V(T_i)} V_t$, $i \in \overline{1,2}$. Let $G_i = G[U_i]$, $i \in \overline{1,2}$.

Then $G = G_1 \cup G_2$, i.e. $V(G) = V(G_1) \cup V(G_2)$
$E(G) = E(G_1) \cup E(G_2)$

by (T1), (T2).

Moreover, $V(G_1 \cap G_2) = V_{t_1} \cap V_{t_2}$ by lemma 8.8 (separation lemma)

$\Rightarrow$ $G_1 \cap G_2$ is complete.

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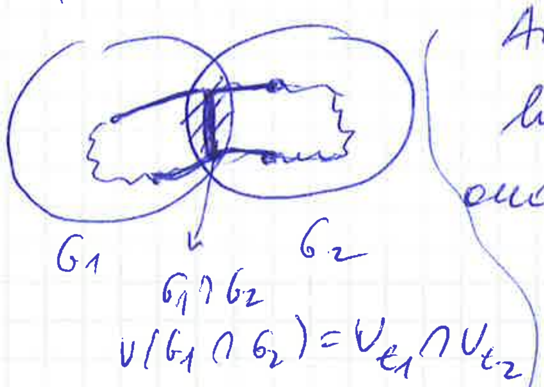
Since $(T_i, \mathcal{V}_i = (V_t)_{t\in T_i})$ is a tree decomposition of G_i in complete parts, $i\in\overline{1,2}$, by the induction hypothesis both G_i, $i\in\overline{1,2}$, are chordal. So any cycle in G_i, $i\in\overline{1,2}$, has a chord.



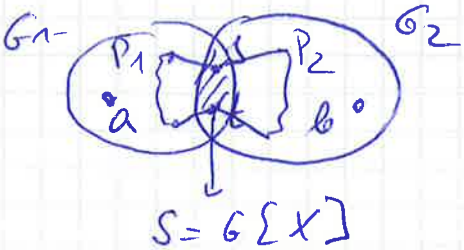
Any cycle C in G with vertices in G_1 and G_2 has to pass through $G_1\cap G_2$ and since $G_1\cap G_2$ is complete the cycle C must have a chord.

$\Longrightarrow$ G is chordal.

$\Longleftarrow$ Assume that G is chordal. If G is complete, there is nothing to show. So assume that G is not complete. Let $a,b\in V(G)$ with $\{a,b\}\notin E(G)$.

Claim: G is the union of smaller chordal graphs G_1, G_2 with $S=G_1\cap G_2$ complete.

Let $X\subseteq V(G)\setminus\{a,b\}$ be a minimal a-b-separator. Let C be the component of $G-X$ containing a. And let $G_1 := G[V(C)\cup X]$ and $G_2 := G[V(G)\setminus C]$



G_i, $i\in\overline{1,2}$, are chordal as subgraphs of G (chordal). So we show $S=G(X]$ is complete. Let $s,t\in X$ and assume $\{s,t\}\notin E(G)$.

Since X is a minimal a-b-separator both s and t have neighbors in C and also in $G-G_1$.

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Let P_i be a shortest path from s to t in G_i, $i \in \overline{1,2}$, with all inner vertices in $G_i \setminus X$.
$\Rightarrow P_1 \cup P_2$ is a chordless cycle of length ≥ 4 ↯ G

end proof of the claim □

Now apply the induction hypotheses on G_i, $i \in \overline{1,2}$.
Let (T_i, V_i) be a tree decomp. of G_i with complete parts, $\forall i \in \overline{1,2}$. By ~~lemma~~ Corollary 8.13 the complete subgraph $S = G_1 \cap G_2$ lies inside one of those parts say in V_{t_i} with $t_i \in V(T_i)$, $i \in \overline{1,2}$.
Then it is easy to check that $(T_1 \cup T_2 \cup \{t_1 t_2\}, V_1 \cup V_2)$ is a tree-decomposition of G into complete parts.
(homework!)

$\boxed{\Sigma}$

Corollary 8.17 For any graph G

$tw(G) = \min \{ \omega(H) - 1 : G \subseteq H, H \text{ is chordal} \}$

Proof $\leq$ By Prop. 8.16 and Corollary 8.13 $\forall H$ chordal there exists a tree-decompos. of width $\omega(H) - 1$. Indeed. Prop. 8.16 $\Rightarrow$ $\exists (T, V)$ with complete parts $\Rightarrow$ every part is a clique in H $\Rightarrow$ $|V_t| \leq \omega(H)$ $\Rightarrow$ width$(T, V) \leq \omega(H) - 1$
On the other hand by Corollary 8.13 any clique is contained in some part
the max clique of G belongs to a part and indeed coincides with a part. Thus $\omega(H) - 1 \leq$ width(T,V)
So width$(T, V) = \omega(H) - 1$. $\Longrightarrow tw(H) = \omega(G) - 1$ (?)
Every such tree-decomp. of some H induces a tree-decomp. of G, i.e. $tw(G) \leq \omega(H) - 1$ $\forall H$ chordal with $G \subseteq H$

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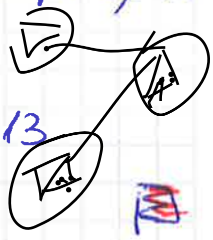
$\Rightarrow$ On the other hand, let us construct an H chordal with $G \subseteq H$ and $tw(G) \geq \omega(H) - 1$. $\geq \min\{\omega(A)-1 : \dots\}$

Let (T, V) be a tree-decomp. of G of width $tw(G)$.

$\forall\, t \in T$ let K_t be the complete graph on V_t. Let H be included $V(H) = V(G)$ and $E(H) = E(G) \cup \bigcup_{t \in V(T)} E(K_t)$. Then (T, V) is also a tree-decomp. of H



By Prop. 8.16 H is chordal and by the Corollary 8.13 $\omega(H) - 1 \leq tw(G)$. $\omega(H) - 1 = tw(H)$

8.6 Tree-width and forbidden minors

Some propositions without proofs.

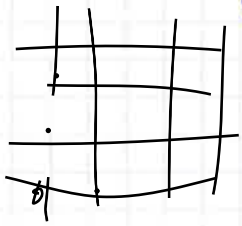
Proposition 8.18 A graph has treewidth < 3 iff it has no K_4 as a minor.

Theorem 8.19 (Robertson and Seymour 1986)

Given a graph H, the graphs without an H minor have bounded treewidth iff H is planar.

Theorem 8.20 (Robertson and Seymour 1986)

$\forall$ natural number r, there exists a natural nr. k s.th. every graph of tree-width at least k has an $r \times r$ grid minor.



8.7 Some facts about tree-decompositions

Def (An equivalent definition of tree-decomposition)

Given a graph $G=(V,E)$, a tree-decomposition of G is a pair $(T, \mathcal{V}=(V_t)_{t\in V(T)})$ such that $V_t \subseteq V(G)$, $\forall t \in V(T)$ and

(T1) $\bigcup_{t\in V(T)} V_t = V(G)$

(T2) $\forall$ $e=\{u,v\}\in E(G)$, $\exists t \in V(T)$, such that $u\in V_t$, $v\in V_t$

(T3') $\forall v\in V(G)$ $T' := \{t\in V(T);\ v\in V_t\}$ (*) is a subtree of T.

V_t (called parts so far) are also called bags.

T3 $\forall t_1,t_2,t_3 \in V(T)$

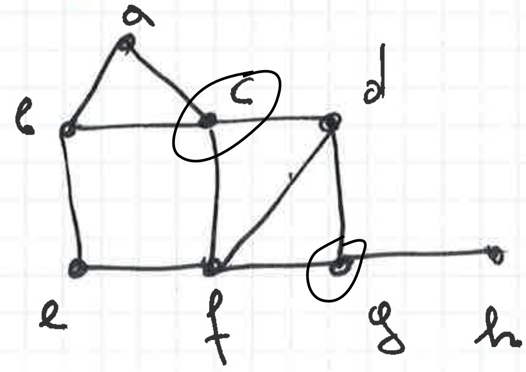
Observe : $T_3' \Leftrightarrow T_3$ is trivial — t_1 t_3 t_2

If $v \in V_{t_1}\cap V_{t_2}$, then by T_3 $v\in V_{t_3}$ $\forall t_3$ in the t_1-t_2-path in T — $V_{t_1}\cap V_{t_2} \subseteq V_{t_3}$

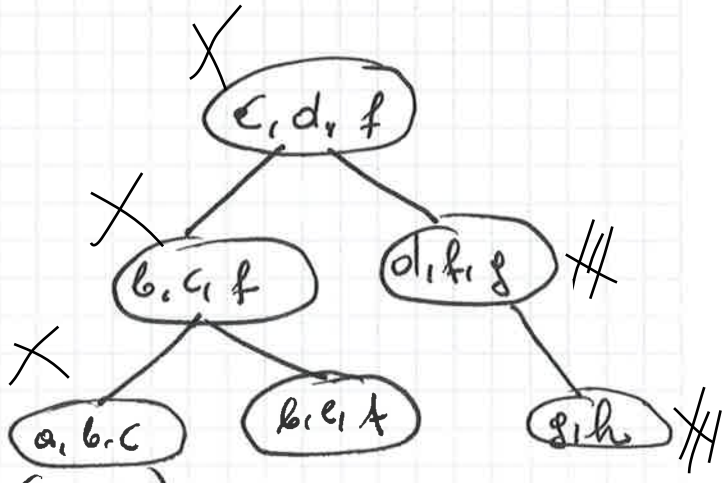
$\Rightarrow$ T' as in (*) is connected in T

Examples

(1)

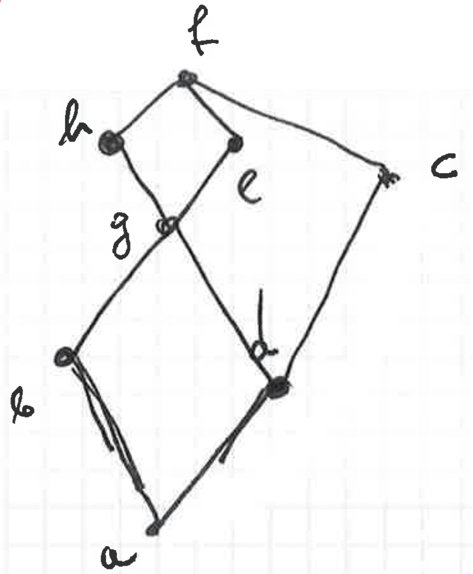


G with $tw(G) \le 2$

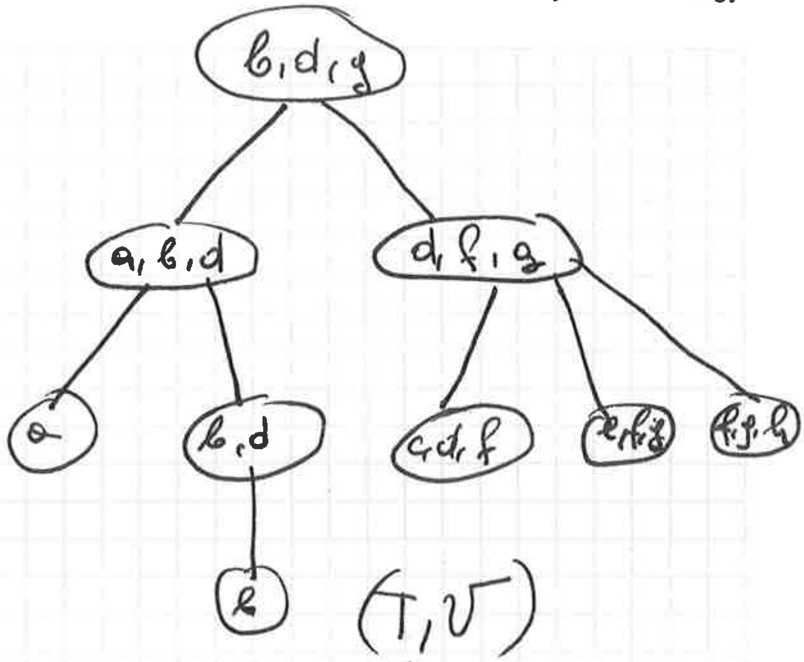


$(T, \mathcal{V})$

(2)



G with $tw(G)=2$



Observation: The tree-width of a graph is equal to the maximum of the tree-widths of its connected components.

Proposition 8.21 Let G be a graph with $E(G) \neq \emptyset$. Then G is a forest iff $tw(G) = 1$.

$V_1 = \{v_1\}$ $V_2 = \{v_1, v_2\}$ t_2 V_1 t_1

Proposition 8.22 Every graph G with $tw(G) = k$ has a vertex $v \in V(G)$ with $\deg(v) \leq k$.

Proof: Let (T,V) be a min. order tree-decomp. of G with width k. $|T|$ Let $\ell \in V(T)$ be a leaf in T and let $(p(\ell),\ell) \in E(T)$. Then $\exists\, v \in V_\ell$ such $v \notin V_{p(\ell)}$ (otherwise we could remove ℓ from T and get a tree-decomp of G with the same width but smaller order). So all neighbors of v are contained in V_ℓ, i.e. $\Gamma(v) \subseteq V_\ell$ and $|V_\ell| \leq k+1 \Rightarrow \deg_G(v) \leq k$.

Let $\{u,v\} \in E(G) \overset{T2}{\Longrightarrow} \exists\, t \in T$ $u, v \in V_t \overset{T3}{\Longrightarrow} T'$ subtree $T' = \{t\}$

$\Longrightarrow \forall v \in V_e$

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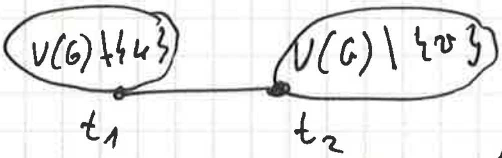
Proposition 8.23 A graph G with $|G| = n$ has tree-width $n-1$ iff G is a clique.

Proof $\Rightarrow$ Assume G is not a clique $\Rightarrow \exists\, u, v \in V(G), \{u,v\} \notin E(G)$.
Construct a $(T, \mathcal{V})$ with $|T| = 2$, $t_1, t_2 \in V(T)$, $\{t_1, t_2\} \in E(T)$



and observe that its width is $n-2$. $\Longrightarrow tw(G) \leq n-2$

$\Leftarrow$ Assume G has a tree decomp. of width $\leq n-2$. Then by Prop. 8.22 $\exists v \in V(G)$ with $deg(v) \leq n-2$
$\Longrightarrow G$ is not a clique.

Proposition 8.24 A graph G has tree-width at most 2 iff G is a subgraph of a series-parallel graph.

(no proof)

Def A multigraph G is called a series-parallel graph if its is obtained from an independent set by using the following operations

(•) Add a new node and connect it to an existing node by an edge

(••) Add a self loop

(•••) Add an edge parallel to an existing edge

(••••) Subdivide an edge by creating a node in the middle

Illustration

serial combination

parallel combination

Proposition 8.25

It is NP-hard to determine the tree-width of a graph.
But there are algorithms that determine whether $tw(G) \le k$ in time $O(n^k)$ where $|G| = n$ and $k \in \mathbb{N}$, k arbitrary.

Proposition 8.25

For a graph G with $n := |V(G)|$ and $tw(G) = k$

(i) a tree-decomp. with width k can be determined in $O(n^{k+O(1)})$ time. Arnborg, Corneil, Proskurowski 1987

(ii) a tree-decomp. with width k can be determined in $2^{\tilde{O}(k^3)} n$ time Bodlaender 1996

(iii) a tree decomp of width $5k+4$ can be determined in $2^{O(k)} n$ time. Bodlaender et al, 2013

(iv) a tree decomposition of width $O(k\sqrt{\log k})$ can be found in polynomial time in n.
Feige, Hajiaghai, Lee 2008

8.7 Solving the weighted max independent set (WMIS) problem on graphs with bounded tree-width

Proposition 8.26 Given a tree-decomposition $(T, \mathcal{V})$ of width w with $n = |T|$ for a graph G, the max. weighted indep. set problem can be solved in $O(2^w \cdot n)$ where $n = |G|$. We sketch a proof.

→ Consider the tree T to be rooted (root arbitrary but fixed)

Notation: $V_x \subseteq V(G)$ part or bag of $x \in V(T)$.

T_x the subtree rooted at x

$$G_x := \bigcup_{z \in T_x} V_z \quad \forall x \in V(T)$$

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→ Generalize the dynamic programming idea used to solve WMIS on trees:

instead of computing 2 values $w^-(v)$ and $w^+(v)$ $\forall v \in V(G)$ we compute $\underline{2^{|V_x|} \leq 2^{w+1}}$ values for each Bag V_x.

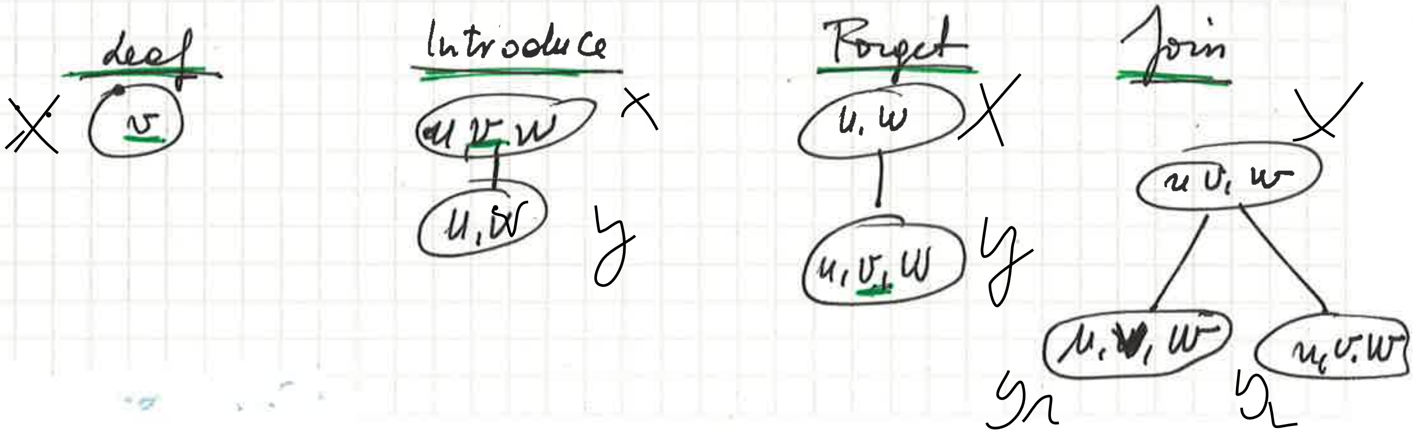
\# Subproblems

Let $\underline{M[x, S]}$ be the max weight of an indep. set I in G_x ($I \subseteq G_x$) such that $I \cap V_x = S$ $\forall x \in V(T)$ / $\forall S \subseteq V_x$

⟹ Question: How to determine $M[x, S]$ if all values are known for the children of x? Goal $\max_{S \subseteq V_r} M[r, S]$

Def A rooted tree-decomposition (T, V) of a graph G is nice if every vertex $x \in V(T)$ is of one of the following 4 types

- Leaf: no children, $|V_x| = 1$
- Introduce: 1 child y, $V_x = V_y \cup \{v\}$ for some $v \in V(G)$
- Forget: 1 child y, $V_x = V_y \setminus \{v\}$ for some $v \in V(G)$
- Join: 2 children y_1, y_2 with $V_x = V_{y_1} = V_{y_2}$



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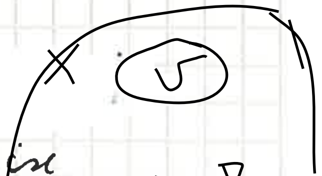
Proposition 8.27

Let G be a graph and $(T,\mathcal{V})$ be a tree-decomp of G with width w and $n=|V(G)|$. $(T,\mathcal{V})$ can be turned into a nice tree decomposition of width w and $O(wn)$ vertices in $O(nw^2)$.

No proof.

Recall our goal: How to determine $M[x,S]$ if all values are known for the children?

- If x is a leaf: $M[x,\{v\}] = \text{weight}(v)$
  and $M[x,S] = -\infty$ otherwise if $S \neq \{v\}$



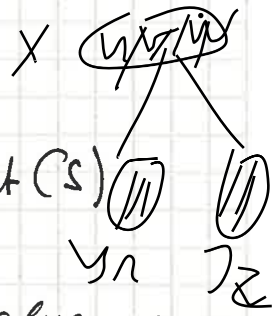
- If x is an introduce:

$$M[x,S] = \begin{cases} M[y,S] & \text{if } v \notin S \\ M[y,S\setminus\{v\}] + \text{weight}(v) & \text{if } v \in S \text{ but } \Gamma(v)\cap S = \emptyset \\ -\infty & \text{if } v\in S \text{ and } \Gamma(v)\cap S \neq \emptyset \end{cases}$$

- If x is a forget: 2 cases $v \in I$ (where I indep. set with weight $M[x,S]$), $v \notin I$

$$M[x,S] = \max\{M[y,S],\ M[y,S\cup\{v\}]\}$$

- If x is a join:



$$M[x,S] = M[y_1,S] + M[y_2,S] - \text{weight}(S)$$

There are at most $2^{w+1}n$ subproblems to solve so as to obtain $M[x,S]$ $\forall x \in V(T)$ $\forall S \subset U_x$ and each problem can be solved in constant time assuming that the children has been solved. Observe that $|V(G)| \le |V(T)|\cdot(w+1) = n(w+1)$.

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Examples of other NP-hard problems which are polynomially solvable in graphs with bounded tree-width.

- weighted min vertex cover
- vertex coloring
- Hamiltonian cycle
- subgraph isomorphism
- list coloring
- equitable coloring . . . .

But there are also NP-hard problems which remain hard also on graphs with bounded tree-width...

see e.g.
K. Meeks and A. Scott,
The parametrized complexity of list problems on graphs of bounded tree width,
arXiv: 1110.4077v3, 4 Aug 2016

A valuable source of literature on discrete maths, optimization (incl. combinatorial opt) (theoretical) computer science and much more maintained by Cornell University
https://arxiv.org

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